

# The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer

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## Abstract

We calculate the massive flavor non-singlet Wilson coefficient for the heavy flavor contributions to the polarized structure function  $g_1(x, Q^2)$  in the asymptotic region  $Q^2 \gg m^2$  to 3-loop order in Quantum Chromodynamics at general values of the Mellin variable  $N$  and the momentum fraction  $x$ , and derive heavy flavor corrections to the Bjorken sum-rule. Numerical results are presented for the charm quark contribution. Results on the structure function  $g_2(x, Q^2)$  in the twist-2 approximation are also given.

# 1 Introduction

Massless and massive contributions to the unpolarized and polarized structure functions in deep-inelastic scattering exhibit different scaling violations. For a precise determination of the QCD scale  $\Lambda_{\text{QCD}}$  or the strong coupling constant  $\alpha_s(M_Z^2)$  their precise knowledge is therefore of importance [1]. In the case of the polarized structure function  $g_1(x, Q^2)$  the complete heavy flavor corrections are only available at 1-loop order [2, 3]<sup>1</sup>. At higher orders in the coupling constant, the heavy flavor contributions were calculated in the asymptotic region  $Q^2 \gg m^2$  based on the factorization derived in Ref. [5]. Here  $Q^2$  denotes the virtuality of the exchanged gauge boson and  $m$  the heavy quark mass. The  $O(\alpha_s^2)$  corrections in the polarized case were calculated in Refs. [6, 7]. In the case of the structure function  $g_1(x, Q^2)$ , the 1-loop heavy flavor corrections have been accounted for at next-to-leading order (NLO) QCD analysis [8]. The corresponding flavor non-singlet corrections in the unpolarized case were calculated for pure photon exchange to  $O(\alpha_s^2)$  in [5, 9] and in Ref. [10] to  $O(\alpha_s^3)$ .

In the present paper we calculate the  $O(\alpha_s^3)$  massive flavor non-singlet Wilson coefficient for the inclusive structure function  $g_1(x, Q^2)$  in the asymptotic region  $Q^2 \gg m^2$ , and also present the corresponding  $O(\alpha_s^2)$  result, extending Refs. [6, 7].

The differential cross section for polarized deep-inelastic scattering [11–13] is given by

$$\frac{d^2\sigma_B}{dx dy} = \frac{2\pi\alpha^2}{Q^4} \lambda_N^p f^p S [S_1^p(x, y)g_1(x, Q^2) + S_2^p(x, y)g_2(x, Q^2)] , \quad (1.1)$$

with

$$\begin{aligned} f^L &= 1, & f^T &= \cos(\beta - \varphi) \frac{d\varphi}{2\pi} \sqrt{\frac{4M^2x}{Sy} \left[ 1 - y - \frac{M^2xy}{S} \right]}, \\ S_1^L &= 2xy \left[ (2 - y) - 2\frac{M^2}{S}xy \right], & S_1^T &= 2xy^2, \\ S_2^L &= -8x^2y\frac{M^2}{S}, & S_2^T &= 4xy. \end{aligned} \quad (1.2)$$

Here  $\alpha = e^2/(4\pi)$  denotes the fine structure constant,  $M$  is the nucleon mass,  $S = (p + l)^2$  is the center of mass energy of the lepton-nucleon system, with  $p$  and  $l$  the nucleon and lepton 4-momenta, respectively,  $q = l - l'$  is the 4-momentum transfer and  $Q^2 = -q^2$ .  $x = Q^2/(2p \cdot q)$  and  $y = p \cdot q / p \cdot l$  are the Bjorken variables.  $\lambda_N^p$  denotes the degree of the nucleon polarization. The spin 4-vectors in the longitudinal and transverse cases are given by

$$S_L = M(0, 0, 0; 1) \quad (1.3)$$

$$S_T = M(0, \cos(\beta), \sin(\beta); 0) , \quad (1.4)$$

and  $\varphi$  denotes the angle between the vectors of the spin and the outgoing lepton. It contributes in a non-trivial way in the case of transverse polarization.

The polarized structure functions are denoted by  $g_1(x, Q^2)$  and  $g_2(x, Q^2)$ . In the leading twist approximation, the heavy flavor contributions to the structure function  $g_1(x, Q^2)$  is given by, cf. [14],

$$g_1(x, Q^2) = \frac{1}{2} \left\{ \sum_{k=1}^{N_F} e_i^k \left\{ L_{q,g_1}^{\text{NS}} \left( x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes [\Delta f_k(x, \mu^2, N_F) + \Delta f_{\bar{k}}(x, \mu^2, N_F)] \right. \right.$$

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<sup>1</sup>For an implementation in Mellin space, see [4].

$$\begin{aligned}
& + \frac{1}{N_F} L_{q,g_1}^{\text{PS}} \left( x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta \Sigma(x, \mu^2, N_F) \\
& + \frac{1}{N_F} L_{g,g_1}^{\text{S}} \left( x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta G(x, \mu^2, N_F) \Big\} \\
& + e_Q^2 \left[ H_{q,g_1}^{\text{PS}} \left( x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta \Sigma(x, \mu^2, N_F) \right. \\
& \left. + H_{g,g_1}^{\text{PS}} \left( x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta G(x, \mu^2, N_F) \right] \Big\} , \tag{1.5}
\end{aligned}$$

with  $\Delta f_{k(\bar{k})}$  the  $N_F$  light flavor polarized (anti)quark densities,  $\Delta G$  and  $\Delta \Sigma = \sum_{l=1}^{N_F} [\Delta f_l + \Delta f_{\bar{l}}]$  the polarized gluon and singlet distributions, and  $e_i$  and  $e_Q$  the electric charges of the light quarks and the heavy quark  $Q$ , respectively.  $\mu$  denotes the factorization scale and  $\otimes$  the Mellin convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) . \tag{1.6}$$

The actual flavor non-singlet distribution is defined by

$$\Delta^{\text{NS}}(x, Q^2) = \sum_{k=1}^{N_F} e_k^2 \left[ \Delta f_k(x, \mu^2, N_F) + \Delta f_{\bar{k}}(x, \mu^2, N_F) - \frac{1}{N_F} \Delta \Sigma(x, \mu^2, N_F) \right] . \tag{1.7}$$

However, according to the representation (1.5), we will consider its whole first term, depending on  $L_{q,1}^{\text{NS}}$  as the non-singlet contribution in what follows. The structure function  $g_2(x, Q^2)$  can be obtained from  $g_1(x, Q^2)$  using the Wandzura-Wilson relation [15].

The paper is organized as follows. In Section 2 we calculate the heavy flavor contributions to the non-singlet Wilson coefficient in the asymptotic region  $Q^2 \gg m^2$  to the structure function  $g_1(x, Q^2)$  to 3-loop order in the strong coupling constant. We present the results both in Mellin  $N$  and  $x$ -space. Numerical results are given in Section 3. Consequences for the polarized Bjorken sum rule are discussed in Section 4, and Section 5 contains the conclusions.

## 2 The Wilson Coefficient

The heavy flavor non-singlet Wilson coefficient contributing to the structure function  $g_1(x, Q^2)$  in the asymptotic region  $Q^2 \gg m^2$  receives its first contributions at  $O(\alpha_s^2)$ . In previous analyses [6,7] the tagged flavor case at  $O(\alpha_s^2)$  has been considered. In what follows we will refer to the inclusive case, i.e. the complete contribution to the structure function  $g_1(x, Q^2)$ , and consider the terms due a single heavy quark.

The non-singlet heavy flavor Wilson coefficient contributing to the structure function  $g_1(x, Q^2)$  in the asymptotic region  $Q^2 \gg m^2$  is given by [16]

$$\begin{aligned}
L_{q,g_1}^{\text{h,NS}}(N_F + 1) &= a_s^2 \left[ A_{qq,Q}^{(2),\text{NS}}(N_F + 1) + \hat{C}_{q,g_1}^{(2),\text{NS}}(N_F) \right] \\
&+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,g_1}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,g_1}^{(3),\text{NS}}(N_F) \right] . \tag{2.1}
\end{aligned}$$

Here  $A_{qq,Q}^{\text{NS}}$  is the massive non-singlet operator matrix element (OME) and the label ‘ $N_F + 1$ ’ symbolically denotes that the OME is calculated at  $N_F$  massless and one massive flavor,  $a_s = \alpha_s/(4\pi) \equiv g_s^2/(4\pi)^2$  parameterizes the strong coupling constant, and we use the convention

$$\hat{f}(N_F) = f(N_F + 1) - f(N_F) . \quad (2.2)$$

The calculation of the different contributions to the Wilson coefficient is performed in  $D = 4 + \varepsilon$  dimensions to regulate the Feynman integrals. In the present polarized case the treatment of  $\gamma_5$  has to be considered. In the flavor non-singlet case both for the massive OMEs and the massless Wilson coefficients  $\gamma_5$  always appears in traces along one massless line and there is a Ward-Takahashi identity which implies the use of anti-commuting  $\gamma_5$ .

The inclusive massive OME  $A_{qq,Q}^{\text{NS}}$  to 3-loop order for even and odd moments  $N$  has been calculated in Ref. [10]. The corresponding diagrams have been reduced using integration-by-parts relations [17] applying an extension of the package **Reduze 2** [18]<sup>2</sup>. The master integrals have been calculated using hypergeometric, Mellin-Barnes and differential equation techniques, mapping them to recurrences, which have been solved by modern summation technologies using extensively the packages **Sigma** [21, 22], **EvaluateMultiSums**, **SumProduction** [23], **ρsum** [24], and **HarmonicSums** [25].

The massless Wilson coefficients  $C_{q,g_1}(x, Q^2)$  from 1- to 3-loop order were calculated in Refs. [26–29]. At 3-loop order those of the structure function  $g_1$  are obtained by that of  $F_3$  [29], setting the  $d_{abc}$  terms in  $\hat{C}_{q,g_1}^{(3),\text{NS}}(N_F)$  to zero, cf. also [30, 31]. The non-singlet OMEs  $A_{qq,Q}^{(k),\text{NS}}$  at 2- and 3-loop order were calculated in [5, 9] and [10], respectively.

For comparison, the massless flavor non-singlet Wilson coefficient in Mellin space is given by [28, 29]

$$L_{q,g_1}^{\text{NS}}(N_F) = 1 + \sum_{k=1}^3 a_s^k C_{q,g_1}^{(k),\text{NS}}(N_F) . \quad (2.3)$$

In Mellin  $N$  space the Wilson coefficient can be expressed by nested harmonic sums  $S_{\vec{a}}(N)$  [32] which are defined by

$$S_{b,\vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z}, b, a_i \neq 0, N > 0, N \in \mathbb{N}. \quad (2.4)$$

In the following, we drop the argument  $N$  of the harmonic sums and use the short-hand notation  $S_{\vec{a}}(N) \equiv S_{\vec{a}}$ . The Wilson coefficients depend on the logarithms

$$L_Q = \ln \left( \frac{Q^2}{\mu^2} \right) \quad \text{and} \quad L_M = \ln \left( \frac{m^2}{\mu^2} \right), \quad (2.5)$$

where the renormalization scale has been set equal to the factorization scale  $\mu = \mu_R = \mu_F$ .

As a short-hand notation we define the leading order splitting function  $\Delta\gamma_{qq}^{(0)}$  up to its color factor

$$\Delta\gamma_{qq}^{(0)} = 4 \left[ 2S_1 - \frac{3N^2 + 3N + 2}{2N(N+1)} \right] . \quad (2.6)$$

The massive Wilson coefficient for the structure function  $g_1(x, Q^2)$  in the asymptotic region in Mellin space in the on-shell scheme is given by

$$L_{q,g_1}^{\text{h,NS}}(N) = \frac{1}{2} [1 - (-1)^N] \left\{ a_s^2 C_F T_F \left\{ -\frac{1}{3} [L_M^2 + L_Q^2] \Delta\gamma_{qq}^{(0)} + L_M \left[ -\frac{2P_1}{9N^2(N+1)^2} - \frac{80}{9} S_1 \right. \right. \right.$$

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<sup>2</sup>The package **Reduze 2** uses the packages **Fermat** [19] and **Ginac** [20].

$$\begin{aligned}
& + \frac{16}{3} S_2 \Big] + L_Q \Big[ -\frac{2P_6}{9N^2(N+1)^2} + \frac{4(29N^2 + 29N - 6)}{9N(N+1)} S_1 + \frac{8}{3} S_1^2 - 8S_2 \Big] + \frac{16}{3} S_{2,1} \\
& + \frac{P_{34}}{27N^3(N+1)^3} + \left( -\frac{2P_{11}}{27N^2(N+1)^2} + \frac{8}{3} S_2 \right) S_1 - \frac{2(29N^2 + 29N - 6)}{9N(N+1)} S_1^2 \\
& - \frac{8}{9} S_1^3 + \frac{2(35N^2 + 35N - 2)}{3N(N+1)} S_2 - \frac{112}{9} S_3 \Big\} \\
& + a_s^3 \left\{ C_{FF}^2 T_F \left[ \frac{1}{6} [L_Q^3 + L_M^2 L_Q] \Delta \gamma_{qq}^{(0)2} + L_M^2 \left[ -\frac{2P_{28}}{3N^3(N+1)^3} - \frac{16}{3} S_1^3 \right. \right. \right. \\
& + \frac{2P_5}{3N^2(N+1)^2} S_1 - \frac{4(N-1)(N+2)}{N(N+1)} S_1^2 + \frac{64}{3} S_3 + \frac{64}{3} S_{-3} - \frac{128}{3} S_{-2,1} \\
& + \left( -\frac{64}{3N(N+1)} + \frac{128}{3} S_1 \right) S_{-2} + \frac{10}{3} \Delta \gamma_{qq}^{(0)} S_2 \Big] + L_Q^2 \left[ -\frac{2P_{30}}{9N^3(N+1)^3} \right. \\
& + \frac{2P_{12}}{9N^2(N+1)^2} S_1 - \frac{4(107N^2 + 107N - 54)}{9N(N+1)} S_1^2 - 16S_1^3 + \frac{64}{3} [S_3 + S_{-3}] \\
& + \left( -\frac{64}{3N(N+1)} + \frac{128}{3} S_1 \right) S_{-2} - \frac{128}{3} S_{-2,1} + \frac{22}{3} \Delta \gamma_{qq}^{(0)} S_2 \Big] \\
& + L_M L_Q \Delta \gamma_{qq}^{(0)} \left[ \frac{P_1}{9N^2(N+1)^2} + \frac{40}{9} S_1 - \frac{8}{3} S_2 \right] + L_M \left[ \frac{P_{40}}{9N^4(N+1)^4} \right. \\
& + \left( \frac{2P_{31}}{9N^3(N+1)^3} + \frac{16(59N^2 + 59N - 6)}{9N(N+1)} S_2 - \frac{256}{3} S_3 - \frac{256}{3} S_{-2,1} \right) S_1 \\
& + \left( -\frac{4P_3}{3N^2(N+1)^2} + \frac{32}{3} S_2 \right) S_1^2 - \frac{160}{9} S_1^3 - \frac{4P_8}{9N^2(N+1)^2} S_2 - 32S_2^2 \\
& + \frac{32(29N^2 + 29N + 12)}{9N(N+1)} S_3 - \frac{256}{3} S_4 + \left( -\frac{64(16N^2 + 10N - 3)}{9N^2(N+1)^2} - \frac{128}{3} S_2 \right. \\
& + \frac{1280}{9} S_1 \Big) S_{-2} + \left( \frac{64(10N^2 + 10N + 3)}{9N(N+1)} - \frac{128}{3} S_1 \right) S_{-3} - \frac{128}{3} S_{-4} \\
& + \frac{128}{3} S_{3,1} - \frac{128(10N^2 + 10N - 3)}{9N(N+1)} S_{-2,1} - \frac{128}{3} S_{-2,2} + \frac{512}{3} S_{-2,1,1} \\
& + 8\Delta \gamma_{qq}^{(0)} \zeta_3 \Big] + L_Q \left[ \frac{4P_{48}}{27N^4(N+1)^4(N+2)^3} + \left( -\frac{4P_{36}}{27N^3(N+1)^3} + \frac{640}{9} S_3 \right. \right. \\
& - \frac{32(67N^2 + 67N - 21)}{9N(N+1)} S_2 + \frac{64}{3} S_{2,1} + \frac{512}{3} S_{-2,1} \Big) S_1 + \left( \frac{2P_{15}}{27N^2(N+1)^2} \right. \\
& - \frac{224}{3} S_2 \Big) S_1^2 + \frac{32(4N-1)(4N+5)}{9N(N+1)} S_1^3 + \frac{80}{9} S_1^4 + \frac{2P_{14}}{9N^2(N+1)^2} S_2 + 48S_2^2 \\
& - \frac{32(53N^2 + 53N + 16)}{9N(N+1)} S_3 + \frac{352}{3} S_4 + \left( -\frac{64P_{27}}{9(N-1)N^2(N+1)^2(N+2)} \right. \\
& - \frac{128(10N^2 + 10N - 3)}{9N(N+1)} S_1 - \frac{256}{3} S_1^2 + \frac{256}{3} S_2 \Big) S_{-2} + 64S_{-2}^2 + \frac{448}{3} S_{-4} \\
& + \left( -\frac{64(10N^2 + 10N + 9)}{9N(N+1)} + \frac{256}{3} S_1 \right) S_{-3} + \frac{16(9N^2 + 9N - 2)}{3N(N+1)} S_{2,1}
\end{aligned}$$

$$\begin{aligned}
& +64S_{3,1} + \frac{128(10N^2 + 10N - 3)}{9N(N+1)}S_{-2,1} - \frac{256}{3}S_{-3,1} - 64S_{2,1,1} \\
& - \frac{512}{3}S_{-2,1,1} + \left( -\frac{16(9N^2 + 9N - 2)}{N(N+1)} + 64S_1 \right) \zeta_3 \Big] + \frac{P_{46}}{162N^5(N+1)^5} \\
& - \frac{128(112N^3 + 112N^2 - 39N + 18)}{81N^2(N+1)}S_{-2,1} + \left( \frac{P_{45}}{162N^4(N+1)^4} - \frac{64}{9}S_2^2 \right. \\
& + \frac{8P_{16}}{81N^2(N+1)^2}S_2 - \frac{8(347N^2 + 347N + 54)}{27N(N+1)}S_3 + \frac{128}{9N(N+1)}S_{2,1} \\
& + \frac{704}{9}S_4 - \frac{320}{9}S_{3,1} - \frac{256(10N^2 + 10N - 3)}{27N(N+1)}S_{-2,1} + \frac{1024}{9}S_{-2,1,1} \\
& \left. - \frac{256}{9}S_{-2,2} \right) S_1 + \left( \frac{P_{25}}{9N^3(N+1)^3} + \frac{16(5N^2 + 5N - 4)}{9N(N+1)}S_2 - \frac{128}{9}S_{2,1} \right. \\
& + 16S_3 - \frac{256}{9}S_{-2,1} \Big) S_1^2 + \left( -\frac{16P_4}{27N^2(N+1)^2} + \frac{128}{27}S_2 \right) S_1^3 + \left( \frac{400}{27}S_3 \right. \\
& + \frac{P_{24}}{81N^3(N+1)^3} + \frac{256}{3}S_{-2,1} \Big) S_2 - \frac{32(23N^2 + 23N - 3)}{27N(N+1)}S_2^2 + \frac{512}{9}S_5 \\
& + \frac{8P_{17}}{81N^2(N+1)^2}S_3 - \frac{176(17N^2 + 17N + 6)}{27N(N+1)}S_4 + \left( -\frac{64P_9}{81N^3(N+1)^3} \right. \\
& + \frac{128P_7}{81N^2(N+1)^2}S_1 - \frac{128}{9N(N+1)}S_1^2 + \frac{256}{27}S_1^3 - \frac{1280}{27}S_2 + \frac{512}{27}S_3 \\
& \left. - \frac{512}{9}S_{2,1} \right) S_{-2} + \left( \frac{64(112N^3 + 224N^2 + 169N + 39)}{81N(N+1)^2} + \frac{128}{9}S_1^2 \right. \\
& + \frac{128}{9}S_2 - \frac{128(10N^2 + 10N + 3)}{27N(N+1)}S_1 \Big) S_{-3} + \left( -\frac{128(10N^2 + 10N + 3)}{27N(N+1)} \right. \\
& + \frac{256}{9}S_1 \Big) S_{-4} + \frac{256}{9}S_{-5} + \frac{16P_2}{9N^2(N+1)^2}S_{2,1} + \frac{256}{9}S_{2,3} - \frac{512}{9}S_{2,-3} \\
& + \frac{16(89N^2 + 89N + 30)}{27N(N+1)}S_{3,1} - \frac{512}{9}S_{4,1} - \frac{128(10N^2 + 10N - 3)}{27N(N+1)}S_{-2,2} \\
& + \frac{512}{9}S_{-2,3} + \frac{512}{9}S_{2,1,-2} + \frac{256}{9}S_{3,1,1} + \frac{512(10N^2 + 10N - 3)}{27N(N+1)}S_{-2,1,1} \\
& + \frac{512}{9}S_{-2,2,1} - \frac{2048}{9}S_{-2,1,1,1} + \frac{16(2N^3 + 2N^2 + 2N + 1)}{3N^3(N+1)^3}\zeta_2 + \left( -\frac{64}{3}S_2 \right. \\
& + \frac{2P_{13}}{9N^2(N+1)^2} - \frac{1208}{9}S_1 \Big) \zeta_3 + \left( \frac{8}{3}S_{2,1,1} - \frac{8}{3}B_4 + 12\zeta_4 \right) \Delta\gamma_{qq}^{(0)} \\
& + (-1)^N \left( -L_M^2 \frac{64}{3(N+1)^3} - L_Q^2 \frac{64}{3(N+1)^3} + L_M \left[ -\frac{256(4N+1)}{9(N+1)^4} \right. \right. \\
& \left. \left. + \frac{128}{3(N+1)^3}S_1 \right] + L_Q \frac{64P_{39}}{9(N-1)N^2(N+1)^4(N+2)^3} + \frac{16P_{41}}{81N^5(N+1)^5} \right. \\
& \left. - \frac{32P_{26}}{27N^4(N+1)^4}S_1 + \frac{64(2N^2 + 2N + 1)}{9N^3(N+1)^3}S_1^2 + \frac{64(2N^2 + 2N + 1)}{9N^3(N+1)^3}S_2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16(2N^3 + 2N^2 + 2N + 1)}{3N^3(N+1)^3} \zeta_2 \Bigg] + C_A C_F T_F \left[ L_M^3 \frac{22}{27} \Delta \gamma_{qq}^{(0)} + L_Q^3 \frac{44}{27} \Delta \gamma_{qq}^{(0)} \right. \\
& + L_M^2 \left[ \frac{2P_{21}}{9N^3(N+1)^2} - \frac{184}{9} S_1 + \left( \frac{32}{3N(N+1)} - \frac{64}{3} S_1 \right) S_{-2} - \frac{32}{3} [S_3 + S_{-3}] \right. \\
& + \left. \frac{64}{3} S_{-2,1} \right] + L_Q^2 \left[ \frac{2P_{23}}{27N^3(N+1)^2} - \frac{16(194N^2 + 194N - 33)}{27N(N+1)} S_1 - \frac{176}{9} S_1^2 \right. \\
& + \frac{176}{3} S_2 - \frac{32}{3} S_3 + \left( \frac{32}{3N(N+1)} - \frac{64}{3} S_1 \right) S_{-2} - \frac{32}{3} S_{-3} + \frac{64}{3} S_{-2,1} \Bigg] \\
& + L_M \left[ \frac{P_{38}}{81N^4(N+1)^3} + \left( -\frac{8P_{29}}{81N^3(N+1)^3} + 32S_3 + \frac{128}{3} S_{-2,1} \right) S_1 \right. \\
& + \frac{1792}{27} S_2 - \frac{16(31N^2 + 31N + 9)}{9N(N+1)} S_3 + \frac{160}{3} S_4 + \left( \frac{32(16N^2 + 10N - 3)}{9N^2(N+1)^2} \right. \\
& - \frac{640}{9} S_1 + \frac{64}{3} S_2 \Bigg) S_{-2} + \left( -\frac{32(10N^2 + 10N + 3)}{9N(N+1)} + \frac{64}{3} S_1 \right) S_{-3} + \frac{64}{3} S_{-4} \\
& - \frac{128}{3} S_{3,1} + \frac{64(10N^2 + 10N - 3)}{9N(N+1)} S_{-2,1} + \frac{64}{3} S_{-2,2} - \frac{256}{3} S_{-2,1,1} - 8\Delta \gamma_{qq}^{(0)} \zeta_3 \Bigg] \\
& + L_Q \left[ -\frac{16(230N^3 + 460N^2 + 213N - 11)}{9N(N+1)^2} S_2 - \frac{4P_{49}}{81N^4(N+1)^4(N+2)^3} \right. \\
& + \left( \frac{4P_{37}}{81N^3(N+1)^3} - \frac{32(11N^2 + 11N + 3)}{9N(N+1)} S_2 - \frac{128}{3} S_{2,1} - \frac{256}{3} S_{-2,1} \right. \\
& + 32S_3 \Bigg) S_1 + \left( \frac{16(194N^2 + 194N - 33)}{27N(N+1)} + \frac{32}{3} S_2 \right) S_1^2 + \frac{352}{27} S_1^3 - \frac{32}{3} S_2^2 \\
& + \frac{16(368N^2 + 368N - 9)}{27N(N+1)} S_3 - \frac{224}{3} S_4 + \left( \frac{32P_{27}}{9(N-1)N^2(N+1)^2(N+2)} \right. \\
& + \frac{64(10N^2 + 10N - 3)}{9N(N+1)} S_1 + \frac{128}{3} S_1^2 - \frac{128}{3} S_2 \Bigg) S_{-2} - 32S_{-2}^2 + \left( -\frac{128}{3} S_1 \right. \\
& + \frac{32(10N^2 + 10N + 9)}{9N(N+1)} \Bigg) S_{-3} - \frac{224}{3} S_{-4} - \frac{64(11N^2 + 11N - 3)}{9N(N+1)} S_{2,1} \\
& - \frac{64}{3} S_{3,1} - \frac{64(10N^2 + 10N - 3)}{9N(N+1)} S_{-2,1} + \frac{128}{3} S_{-3,1} + 64S_{2,1,1} + \frac{256}{3} S_{-2,1,1} \\
& + \left( 96 - 64S_1 \right) \zeta_3 \Bigg] + \frac{64(112N^3 + 112N^2 - 39N + 18)}{81N^2(N+1)} S_{-2,1} \\
& + \frac{P_{47}}{729N^5(N+1)^5} + \left( -\frac{16(N-1)(2N^3 - N^2 - N - 2)}{9N^2(N+1)^2} S_2 + \frac{112}{9} S_2^2 \right. \\
& - \frac{4P_{44}}{729N^4(N+1)^4} + \frac{80(2N+1)^2}{9N(N+1)} S_3 - \frac{208}{9} S_4 - \frac{8(9N^2 + 9N + 16)}{9N(N+1)} S_{2,1} \\
& + \frac{64}{3} S_{3,1} + \frac{128(10N^2 + 10N - 3)}{27N(N+1)} S_{-2,1} + \frac{128}{9} S_{-2,2} - \frac{512}{9} S_{-2,1,1} \Bigg) S_1 \\
& + \left( \frac{4P_{18}}{9N^3(N+1)^3} + \frac{32}{9N(N+1)} S_2 - \frac{80}{9} S_3 + \frac{128}{9} S_{2,1} + \frac{128}{9} S_{-2,1} \right) S_1^2
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{4P_{35}}{81N^3(N+1)^3} + \frac{496}{27}S_3 - \frac{64}{3}S_{2,1} - \frac{128}{3}S_{-2,1} \right) S_2 - \frac{64}{27}S_1^3 S_2 \\
& - \frac{4(15N^2 + 15N + 14)}{9N(N+1)}S_2^2 - \frac{8P_{20}}{81N^2(N+1)^2}S_3 + \frac{4(443N^2 + 443N + 78)}{27N(N+1)}S_4 \\
& - \frac{224}{9}S_5 + \left( \frac{32P_9}{81N^3(N+1)^3} - \frac{64P_7}{81N^2(N+1)^2}S_1 + \frac{64}{9N(N+1)}S_1^2 - \frac{128}{27}S_1^3 \right. \\
& + \frac{640}{27}S_2 - \frac{256}{27}S_3 + \frac{256}{9}S_{2,1} \left. \right) S_{-2} + \left( -\frac{32(112N^3 + 224N^2 + 169N + 39)}{81N(N+1)^2} \right. \\
& + \frac{64(10N^2 + 10N + 3)}{27N(N+1)}S_1 - \frac{64}{9}S_1^2 - \frac{64}{9}S_2 \left. \right) S_{-3} + \left( \frac{64(10N^2 + 10N + 3)}{27N(N+1)} \right. \\
& - \frac{128}{9}S_1 \left. \right) S_{-4} - \frac{128}{9}S_{-5} - \frac{8P_{19}}{9N^2(N+1)^2}S_{2,1} - \frac{8(13N+4)(13N+9)}{27N(N+1)}S_{3,1} \\
& + \frac{256}{9}[S_{2,-3} + S_{4,1} - S_{-2,3} - S_{2,1,-2} - S_{3,1,1} - S_{-2,2,1}] - \frac{128}{3}S_{2,3} \\
& + \frac{64}{3}S_{2,2,1} + \frac{64(10N^2 + 10N - 3)}{27N(N+1)}S_{-2,2} - \frac{256(10N^2 + 10N - 3)}{27N(N+1)}S_{-2,1,1} \\
& + \frac{224}{9}S_{2,1,1,1} + \frac{1024}{9}S_{-2,1,1,1} - \frac{8(2N^3 + 2N^2 + 2N + 1)}{3N^3(N+1)^3}\zeta_2 \\
& + \left( \frac{P_{22}}{27N^2(N+1)^2} + \frac{4(593N^2 + 593N + 108)}{27N(N+1)}S_1 - 16S_1^2 + 16S_2 \right) \zeta_3 \\
& + \left( \frac{4B_4}{3} - 4S_{2,1,1} - 12\zeta_4 \right) \Delta\gamma_{qq}^{(0)} + (-1)^N \left( L_M^2 \frac{32}{3(N+1)^3} + L_Q^2 \frac{32}{3(N+1)^3} \right. \\
& + L_M \left[ \frac{128(4N+1)}{9(N+1)^4} - \frac{64}{3(N+1)^3}S_1 \right] - L_Q \frac{32P_{39}}{9(N-1)N^2(N+1)^4(N+2)^3} \\
& - \frac{8P_{41}}{81N^5(N+1)^5} + \frac{16P_{26}}{27N^4(N+1)^4}S_1 - \frac{32(2N^2 + 2N + 1)}{9N^3(N+1)^3}[S_1^2 + S_2] \\
& \left. - \frac{8(2N^3 + 2N^2 + 2N + 1)}{3N^3(N+1)^3}\zeta_2 \right) \Big] + C_F T_F^2 \left[ -L_M^3 \frac{16}{27}\Delta\gamma_{qq}^{(0)} - L_Q^3 \frac{8}{27}\Delta\gamma_{qq}^{(0)} \right. \\
& + L_M^2 \left[ -\frac{8P_1}{27N^2(N+1)^2} - \frac{320}{27}S_1 + \frac{64}{9}S_2 \right] + L_Q^2 \left[ \frac{16(29N^2 + 29N - 6)}{27N(N+1)}S_1 \right. \\
& - \frac{8P_6}{27N^2(N+1)^2} + \frac{32}{9}S_1^2 - \frac{32}{3}S_2 \left. \right] - L_M \frac{248}{81}\Delta\gamma_{qq}^{(0)} + L_Q \left[ \frac{8P_{33}}{81N^3(N+1)^3} \right. \\
& + \left( -\frac{16P_{10}}{81N^2(N+1)^2} + \frac{64}{9}S_2 \right) S_1 - \frac{16(29N^2 + 29N - 6)}{27N(N+1)}S_1^2 - \frac{64}{27}S_1^3 \\
& + \frac{16(35N^2 + 35N - 2)}{9N(N+1)}S_2 - \frac{896}{27}S_3 + \frac{128}{9}S_{2,1} \left. \right] - \frac{2P_{43}}{729N^4(N+1)^4} + \frac{64}{81}S_2 \\
& + \frac{12064}{729}S_1 + \frac{320}{81}S_3 - \frac{64}{27}S_4 - \frac{112}{27}\Delta\gamma_{qq}^{(0)}\zeta_3 \Big] + C_F N_F T_F^2 \left[ -L_M^3 \frac{8}{27}\Delta\gamma_{qq}^{(0)} \right. \\
& - L_Q^3 \frac{16}{27}\Delta\gamma_{qq}^{(0)} + L_M \left[ \frac{4P_{32}}{81N^3(N+1)^3} - \frac{2176}{81}S_1 - \frac{320}{27}S_2 + \frac{64}{9}S_3 \right]
\end{aligned}$$



$$\begin{aligned}
& +L_Q^2 \left[ -\frac{16P_6}{27N^2(N+1)^2} + \frac{32(29N^2+29N-6)}{27N(N+1)}S_1 + \frac{64}{9}S_1^2 - \frac{64}{3}S_2 \right] \\
& +L_Q \left[ \left( -\frac{32P_{10}}{81N^2(N+1)^2} + \frac{128}{9}S_2 \right) S_1 - \frac{32(29N^2+29N-6)}{27N(N+1)}S_1^2 - \frac{128}{27}S_1^3 \right. \\
& + \frac{16P_{33}}{81N^3(N+1)^3} + \frac{32(35N^2+35N-2)}{9N(N+1)}S_2 - \frac{1792}{27}S_3 + \frac{256}{9}S_{2,1} \left. \right] \\
& + \frac{4P_{42}}{729N^4(N+1)^4} - \frac{24064}{729}S_1 + \frac{128}{81}S_2 + \frac{640}{81}S_3 - \frac{128}{27}S_4 + \frac{64}{27}\Delta\gamma_{qq}^{(0)}\zeta_3 \left. \right] \\
& + \hat{C}_{q,g_1}^{\text{NS},(3)}(N_F) \left. \right\} \left. \right\} . \tag{2.7}
\end{aligned}$$

Here the color factors are given by  $C_A = N_c$ ,  $C_F = (N_c^2 - 1)/(2N_c)$ ,  $T_F = 1/2$  in  $SU(N_c)$ , and  $N_c = 3$  in the case of Quantum Chromodynamics.  $\hat{C}_{q,g_1}^{\text{NS},(3)}(N_F)$  denotes the massless Wilson coefficient at 3-loop order, cf. (2.2), and the polynomials  $P_i$  are given by

$$P_1 = -3N^4 - 6N^3 - 47N^2 - 20N + 12 \tag{2.8}$$

$$P_2 = 7N^4 + 14N^3 + 3N^2 - 4N - 4 \tag{2.9}$$

$$P_3 = 19N^4 + 38N^3 - 9N^2 - 20N + 4 \tag{2.10}$$

$$P_4 = 28N^4 + 56N^3 + 28N^2 + 2N + 1 \tag{2.11}$$

$$P_5 = 33N^4 + 54N^3 + 9N^2 - 52N - 28 \tag{2.12}$$

$$P_6 = 57N^4 + 96N^3 + 65N^2 - 10N - 24 \tag{2.13}$$

$$P_7 = 112N^4 + 224N^3 + 121N^2 + 9N + 9 \tag{2.14}$$

$$P_8 = 141N^4 + 246N^3 + 241N^2 - 8N - 84 \tag{2.15}$$

$$P_9 = 181N^4 + 266N^3 + 82N^2 - 3N + 18 \tag{2.16}$$

$$P_{10} = 235N^4 + 524N^3 + 211N^2 + 30N + 72 \tag{2.17}$$

$$P_{11} = 359N^4 + 772N^3 + 335N^2 + 30N + 72 \tag{2.18}$$

$$P_{12} = 501N^4 + 894N^3 + 541N^2 - 116N - 204 \tag{2.19}$$

$$P_{13} = 561N^4 + 1122N^3 + 767N^2 + 302N + 48 \tag{2.20}$$

$$P_{14} = 1131N^4 + 2118N^3 + 1307N^2 + 32N - 276 \tag{2.21}$$

$$P_{15} = 1139N^4 + 2710N^3 + 635N^2 + 216N + 828 \tag{2.22}$$

$$P_{16} = 1199N^4 + 2398N^3 + 1181N^2 + 18N + 90 \tag{2.23}$$

$$P_{17} = 1220N^4 + 2359N^3 + 1934N^2 + 357N - 138 \tag{2.24}$$

$$P_{18} = 3N^5 + 11N^4 + 10N^3 + 3N^2 + 7N + 8 \tag{2.25}$$

$$P_{19} = 12N^5 + 16N^4 + 18N^3 - 15N^2 - 5N - 8 \tag{2.26}$$

$$P_{20} = 27N^5 + 863N^4 + 1573N^3 + 1151N^2 + 144N - 36 \tag{2.27}$$

$$P_{21} = 51N^5 + 102N^4 + 121N^3 + 118N^2 + 48N + 48 \tag{2.28}$$

$$P_{22} = 648N^5 - 2103N^4 - 4278N^3 - 3505N^2 - 682N - 432 \tag{2.29}$$

$$P_{23} = 1407N^5 + 2418N^4 + 1793N^3 + 134N^2 - 384N + 144 \tag{2.30}$$

$$P_{24} = -11145N^6 - 32355N^5 - 37523N^4 - 14329N^3 + 2392N^2 + 120N - 1512 \tag{2.31}$$

$$P_{25} = -151N^6 - 469N^5 - 181N^4 + 305N^3 + 208N^2 + 40N + 8 \tag{2.32}$$

$$P_{26} = 3N^6 + 9N^5 + 70N^4 + 77N^3 + 39N^2 - 10N - 12 \tag{2.33}$$

$$P_{27} = 6N^6 + 18N^5 - N^4 - 20N^3 + 46N^2 + 29N - 6 \tag{2.34}$$

$$P_{28} = 15N^6 + 36N^5 + 30N^4 + 8N^3 + 3N^2 + 16N + 20 \quad (2.35)$$

$$P_{29} = 155N^6 + 465N^5 + 465N^4 + 371N^3 + 108N^2 + 108N + 54 \quad (2.36)$$

$$P_{30} = 216N^6 + 567N^5 + 687N^4 + 381N^3 + 37N^2 - 44N + 12 \quad (2.37)$$

$$P_{31} = 309N^6 + 807N^5 + 693N^4 - 271N^3 - 638N^2 + 68N + 216 \quad (2.38)$$

$$P_{32} = 525N^6 + 1575N^5 + 1535N^4 + 973N^3 + 536N^2 + 48N - 72 \quad (2.39)$$

$$P_{33} = 609N^6 + 1485N^5 + 1393N^4 + 83N^3 - 422N^2 + 156N + 216 \quad (2.40)$$

$$P_{34} = 795N^6 + 2043N^5 + 2075N^4 + 517N^3 - 298N^2 + 156N + 216 \quad (2.41)$$

$$P_{35} = 868N^6 + 2469N^5 + 2487N^4 + 940N^3 + 27N^2 + 63N + 72 \quad (2.42)$$

$$P_{36} = 1770N^6 + 4671N^5 + 4765N^4 + 1205N^3 - 227N^2 + 1044N + 756 \quad (2.43)$$

$$P_{37} = 7531N^6 + 23673N^5 + 23055N^4 + 7375N^3 + 1614N^2 + 936N - 324 \quad (2.44)$$

$$P_{38} = -4785N^7 - 14355N^6 - 4399N^5 + 10327N^4 + 3548N^3 + 3000N^2 + 1080N - 1728 \quad (2.45)$$

$$P_{39} = 25N^7 + 138N^6 + 311N^5 + 464N^4 + 672N^3 + 670N^2 + 264N + 48 \quad (2.46)$$

$$P_{40} = -45N^8 - 162N^7 - 858N^6 - 1960N^5 - 1885N^4 - 1094N^3 - 804N^2 - 40N + 192 \quad (2.47)$$

$$P_{41} = 39N^8 + 138N^7 + 847N^6 + 1371N^5 + 1283N^4 + 485N^3 + 101N^2 + 132N + 72 \quad (2.48)$$

$$P_{42} = 3549N^8 + 14196N^7 + 23870N^6 + 25380N^5 + 15165N^4 + 1712N^3 - 2016N^2 + 144N + 432 \quad (2.49)$$

$$P_{43} = 5487N^8 + 21948N^7 + 36370N^6 + 28836N^5 + 11943N^4 + 4312N^3 + 2016N^2 - 144N - 432 \quad (2.50)$$

$$P_{44} = 10807N^8 + 43228N^7 + 62898N^6 + 39178N^5 + 7027N^4 + 702N^3 + 3240N^2 + 3456N + 1620 \quad (2.51)$$

$$P_{45} = 42591N^8 + 166764N^7 + 245088N^6 + 128254N^5 - 26735N^4 - 40762N^3 - 3928N^2 - 1272N - 2160 \quad (2.52)$$

$$P_{46} = -18351N^{10} - 89784N^9 - 208773N^8 - 267222N^7 - 192265N^6 - 46700N^5 + 14565N^4 + 7730N^3 + 1240N^2 + 1464N + 144 \quad (2.53)$$

$$P_{47} = 165N^{10} + 825N^9 + 106856N^8 + 321746N^7 + 396657N^6 + 247433N^5 + 126914N^4 + 51804N^3 + 6336N^2 + 4752N + 5184 \quad (2.54)$$

$$P_{48} = 828N^{11} + 7632N^{10} + 29217N^9 + 59592N^8 + 66844N^7 + 35738N^6 + 7405N^5 + 16688N^4 + 27880N^3 + 11552N^2 - 3312N - 2304 \quad (2.55)$$

$$P_{49} = 8274N^{11} + 78519N^{10} + 313841N^9 + 686295N^8 + 881001N^7 + 638778N^6 + 204948N^5 + 7992N^4 + 32296N^3 + 26544N^2 - 10656N - 8640 . \quad (2.56)$$

We would like to note that we disagree with the  $O(a_s^2 \ln(Q^2/\mu^2))$  terms given in [28], but agree with the representation in [29, 51].

One obtains the analytic continuation of the harmonic sums to complex values of  $N$  by performing their asymptotic expansion analytically, cf. [33, 34].<sup>3</sup> Furthermore, the nested harmonic sums obey the shift relations

$$S_{b,\bar{a}}(N) = S_{b,\bar{a}}(N-1) + \frac{\text{sign}(b)^N}{N^{|\bar{a}|}} S_{\bar{a}}(N) , \quad (2.57)$$

---

<sup>3</sup>These expansions can now be obtained automatically using the package `HarmonicSums` [25].

through which any regular point in the complex plane can be reached using the analytic asymptotic representation as input. The poles of the nested harmonic sums  $S_{\vec{a}}(N)$  are located at the non-positive integers. In data analyses, one may thus encode the QCD evolution [35] together with the Wilson coefficient for complex values of  $N$  analytically and finally perform one numerical contour integral around the singularities of the problem.<sup>4</sup>

In  $x$ -space the Wilson coefficient is represented in terms of harmonic polylogarithms [37] over the alphabet  $\{f_0, f_1, f_{-1}\}$ , which were again reduced applying the shuffle relations [38]. They are defined by

$$H_{b,\vec{a}}(x) = \int_0^x dy f_b(y) H_{\vec{a}}(y), \quad H_{\underbrace{0,\dots,0}_k}(x) = \frac{1}{k!} \ln^k(x), \quad H_{\emptyset} = 1, \quad (2.58)$$

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x}. \quad (2.59)$$

The Wilson coefficient is represented by three contributions, the  $(\dots)_+$ -function term, the  $\delta(1-x)$ -term, and the regular term. Here the  $+$ -distribution is defined by

$$\int_0^1 dy [F(y)]_+ g(y) = \int_0^1 dy F(y) [g(y) - g(1)]. \quad (2.60)$$

One obtains

$$\begin{aligned} L_{q,g_1}^{\text{NS}}(x) = & a_s^2 \left\{ \left( \frac{1}{1-x} C_F T_F \left[ \frac{8}{3} [L_Q^2 + L_M^2] + L_M \left[ \frac{80}{9} + \frac{16}{3} H_0 \right] + L_Q \left[ -\frac{116}{9} - \frac{32}{3} H_0 \right. \right. \right. \right. \\ & \left. \left. \left. - \frac{16}{3} H_1 \right] + \frac{718}{27} + \frac{268}{9} H_0 + 8 H_0^2 + \left( \frac{116}{9} + \frac{16}{3} H_0 \right) H_1 + \frac{8}{3} H_1^2 + \frac{16}{3} H_{0,1} \right. \right. \\ & \left. \left. \left. - \frac{32}{3} \zeta_2 \right] \right) + \delta(1-x) \left( C_F T_F \left[ 2 [L_M^2 + L_Q^2] + L_M \frac{2}{3} - L_Q \frac{38}{3} + \frac{265}{9} \right] \right) \right. \\ & \left. + C_F T_F \left[ -\frac{4}{3} (x+1) [L_Q^2 + L_M^2] + L_M \left[ -\frac{8}{9} (11x-1) - \frac{8}{3} (x+1) H_0 \right] \right. \right. \\ & \left. \left. + L_Q \left[ \frac{8}{9} (14x+5) + \frac{16}{3} (x+1) H_0 + \frac{8}{3} (x+1) H_1 \right] - \frac{4}{27} (218x+47) \right. \right. \\ & \left. \left. - \frac{8}{9} (28x+13) H_0 - 4(x+1) H_0^2 + \left( -\frac{8}{9} (14x+5) - \frac{8}{3} (x+1) H_0 \right) H_1 \right. \right. \\ & \left. \left. \left. - \frac{4}{3} (x+1) H_1^2 - \frac{8}{3} (x+1) H_{0,1} + \frac{16}{3} (x+1) \zeta_2 \right] \right\} \\ & + a_s^3 \left\{ \left( \frac{1}{(1-x)^2} C_A C_F T_F \left[ -\frac{4}{81} (800x-773) H_0^2 + \frac{32}{81} (94x-121) \zeta_2 \right. \right. \right. \\ & \left. \left. \left. + \frac{32}{9} (x+2) H_{0,1} \right] + \frac{1}{1-x} \left( C_A C_F T_F \left[ -L_M^3 \frac{176}{27} - L_Q^3 \frac{352}{27} + L_M^2 \left[ \frac{184}{9} \right. \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{16}{3} H_0^2 - \frac{32}{3} \zeta_2 \right] + L_Q^2 \left[ \frac{3104}{27} + \frac{704}{9} H_0 + \frac{16}{3} H_0^2 + \frac{352}{9} H_1 - \frac{32}{3} \zeta_2 \right] \right) \right\} \end{aligned}$$

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<sup>4</sup>For precise numerical implementations of the analytic continuation of harmonic sums see [36].

$$\begin{aligned}
& +L_M \left[ \frac{1240}{81} + \frac{1792}{27} H_0 + \frac{248}{9} H_0^2 + \frac{32}{9} H_0^3 - 16 H_0^2 H_1 + 32 H_0 H_{0,1} - \frac{64}{3} H_{0,0,1} \right. \\
& + \left. \left( -\frac{320}{9} - \frac{64}{3} H_0 \right) \zeta_2 + 96 \zeta_3 \right] + L_Q \left[ -\frac{80}{9} H_0^3 - \frac{30124}{81} - \frac{14144}{27} H_0 - \frac{1216}{9} H_0^2 \right. \\
& + \left. \left( -\frac{6208}{27} - \frac{704}{9} H_0 - \frac{16}{3} H_0^2 \right) H_1 + \left( -\frac{352}{9} + \frac{32}{3} H_0 \right) H_1^2 - 64 H_0 H_{0,-1} \right. \\
& + \left. \left( -\frac{704}{9} + \frac{32}{3} H_0 - \frac{128}{3} H_1 \right) H_{0,1} - \frac{128}{3} H_{0,0,1} + 128 H_{0,0,-1} + 64 H_{0,1,1} \right. \\
& + \left. \left( 192 + \frac{128}{3} H_0 + 64 H_1 \right) \zeta_2 - \frac{256}{3} \zeta_3 \right] + \frac{43228}{729} + \frac{3256}{81} H_0 + \frac{496}{81} H_0^3 + \frac{16}{27} H_0^4 \\
& + \left( \frac{32}{3} - \frac{32}{9} H_0 - \frac{160}{9} H_0^2 - \frac{112}{27} H_0^3 \right) H_1 + \frac{8}{9} H_0^2 H_1^2 - \frac{64}{27} H_0 H_1^3 + \left( \frac{368}{9} H_0 \right. \\
& + \frac{16}{3} H_0^2 + \left. \left( -8 - \frac{128}{9} H_0 \right) H_1 + \frac{128}{9} H_1^2 \right) H_{0,1} - \frac{32}{9} H_{0,1}^2 + \left( -\frac{1072}{27} + \frac{32}{9} H_0 \right. \\
& + \frac{320}{9} H_1 \left. \right) H_{0,0,1} + \frac{224}{9} [H_{0,1,1,1} - H_{0,0,1,1}] + \left( \frac{160}{9} H_0 - 32 H_1 + 24 \right) H_{0,1,1} \\
& + \left( -\frac{496}{27} H_0 - \frac{112}{9} H_0^2 + \left( 8 - \frac{160}{9} H_0 \right) H_1 - \frac{128}{9} H_1^2 + \frac{32}{9} H_{0,1} \right) \zeta_2 + \left( -\frac{1196}{27} \right. \\
& + \frac{160}{9} H_0 - \frac{32}{9} H_1 \left. \right) \zeta_3 + \frac{296}{3} \zeta_4 - \frac{32}{3} B_4 \Big] + C_F^2 T_F \left[ L_Q^3 \left[ 16 - \frac{32}{3} H_0 - \frac{64}{3} H_1 \right] \right. \\
& + L_M^2 L_Q \left[ 16 - \frac{32}{3} H_0 - \frac{64}{3} H_1 \right] + L_M^2 \left[ -22 - 16 H_0 + \left( 8 + \frac{128}{3} H_0 \right) H_1 \right. \\
& + \frac{16}{3} H_0^2 + 16 H_1^2 - \frac{64}{3} \zeta_2 \Big] + L_Q^2 \left[ -\frac{334}{3} + \frac{32}{9} H_0 + \left( \frac{856}{9} + \frac{320}{3} H_0 \right) H_1 \right. \\
& + \frac{80}{3} H_0^2 + 48 H_1^2 - \frac{256}{3} \zeta_2 \Big] + L_M L_Q \left[ \frac{88}{3} - \frac{176}{9} H_0 + \left( -\frac{640}{9} - \frac{64}{3} H_0 \right) H_1 \right. \\
& - \frac{32}{3} H_0^2 + \frac{64}{3} \zeta_2 \Big] + L_M \left[ -\frac{206}{3} - \frac{112}{3} H_0 + \frac{88}{9} H_0^2 + \left( \frac{160}{3} + \frac{32}{3} H_0 \right) H_1^2 \right. \\
& + \frac{64}{9} H_0^3 + \left( \frac{152}{3} + \frac{1424}{9} H_0 + \frac{160}{3} H_0^2 \right) H_1 - \frac{64}{3} H_0 H_{0,1} + \left( -\frac{784}{9} - \frac{128}{3} H_0 \right. \\
& - \frac{64}{3} H_1 \left. \right) \zeta_2 - 64 \zeta_3 \Big] + L_Q \left[ \frac{2360}{9} + \frac{4508}{27} H_0 - \frac{160}{3} H_0^2 - \frac{224}{9} H_0^3 - \frac{320}{9} H_1^3 \right. \\
& + \left( -\frac{4556}{27} - \frac{3680}{9} H_0 - 128 H_0^2 \right) H_1 + \left( -\frac{512}{3} - 128 H_0 \right) H_1^2 + 128 H_0 H_{0,-1} \\
& + \left( 48 - \frac{64}{3} H_0 + \frac{64}{3} H_1 \right) H_{0,1} - 256 H_{0,0,-1} - 64 H_{0,1,1} + \left( \frac{2608}{9} + \frac{832}{3} H_0 \right. \\
& + \frac{448}{3} H_1 \left. \right) \zeta_2 + 320 \zeta_3 \Big] - \frac{14197}{54} - \frac{3262}{27} H_0 + \frac{4}{3} H_0^4 + \frac{196}{27} H_0^2 + \frac{380}{81} H_0^3 \\
& + \left( \frac{302}{9} + \frac{13624}{81} H_0 + \frac{1628}{27} H_0^2 + \frac{304}{27} H_0^3 \right) H_1 + \left( \frac{448}{9} + \frac{80}{9} H_0 - \frac{8}{9} H_0^2 \right) H_1^2 \\
& + \frac{128}{27} H_0 H_1^3 + \left( \frac{112}{9} - \frac{1304}{27} H_0 - \frac{32}{3} H_0^2 + \frac{160}{9} H_0 H_1 - \frac{128}{9} H_1^2 \right) H_{0,1}
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{1184}{27} + \frac{128}{9}H_0 - \frac{256}{9}H_1 \right) H_{0,0,1} + \left( -16 - \frac{32}{3}H_0 + \frac{64}{3}H_1 \right) H_{0,1,1} \\
& - \frac{16}{9}H_{0,1}^2 - \frac{128}{9}H_{0,0,0,1} + \frac{64}{3}H_{0,0,1,1} + \left( -\frac{2488}{27} - \frac{1192}{27}H_0 - \frac{80}{9}H_0^2 + \left( -\frac{160}{9} \right. \right. \\
& \left. \left. + \frac{32}{3}H_0 \right) H_1 + \frac{128}{9}H_1^2 - \frac{32}{9}H_{0,1} \right) \zeta_2 + \left( \frac{3088}{27} - \frac{128}{9}H_0 + \frac{160}{9}H_1 \right) \zeta_3 + \frac{64}{3}B_4 \\
& - \frac{664}{9}\zeta_4 \Big] + C_F T_F^2 \left[ L_M^3 \frac{128}{27} + L_Q^3 \frac{64}{27} + L_M^2 \left[ \frac{320}{27} + \frac{64}{9}H_0 \right] + L_Q^2 \left[ -\frac{464}{27} \right. \right. \\
& \left. \left. - \frac{128}{9}H_0 - \frac{64}{9}H_1 \right] + L_M \frac{1984}{81} + L_Q \left[ \frac{3760}{81} + \frac{2144}{27}H_0 + \left( \frac{928}{27} + \frac{128}{9}H_0 \right) H_1 \right. \right. \\
& \left. \left. + \frac{64}{3}H_0^2 + \frac{64}{9}H_1^2 + \frac{128}{9}H_{0,1} - \frac{256}{9}\zeta_2 \right] - \frac{12064}{729} + \frac{64}{81}H_0 - \frac{160}{81}H_0^2 - \frac{32}{81}H_0^3 \right. \\
& \left. + \frac{896}{27}\zeta_3 \right] + C_F N_F T_F^2 \left[ L_M^3 \frac{64}{27} + L_Q^3 \frac{128}{27} + L_Q^2 \left[ -\frac{928}{27} - \frac{256}{9}H_0 - \frac{128}{9}H_1 \right] \right. \\
& \left. + L_M \left[ \frac{2176}{81} - \frac{320}{27}H_0 - \frac{32}{9}H_0^2 \right] + L_Q \left[ \frac{7520}{81} + \frac{4288}{27}H_0 + \frac{128}{3}H_0^2 + \left( \frac{1856}{27} \right. \right. \right. \\
& \left. \left. + \frac{256}{9}H_0 \right) H_1 + \frac{128}{9}H_1^2 + \frac{256}{9}H_{0,1} - \frac{512}{9}\zeta_2 \right] + \frac{24064}{729} + \frac{128}{81}H_0 - \frac{320}{81}H_0^2 \\
& \left. - \frac{64}{81}H_0^3 - \frac{512}{27}\zeta_3 \right] \Big) + \delta(1-x) \left( C_A C_F T_F \left[ -L_M^3 \frac{44}{9} - L_Q^3 \frac{88}{9} + L_M^2 \left[ \frac{34}{3} \right. \right. \right. \\
& \left. \left. - \frac{16}{3}\zeta_3 \right] + L_Q^2 \left[ \frac{938}{9} - \frac{16}{3}\zeta_3 \right] + L_M \left[ -\frac{1595}{27} + \frac{272}{9}\zeta_3 + \frac{68}{3}\zeta_4 \right] + L_Q \left[ -\frac{11032}{27} \right. \right. \\
& \left. \left. - \frac{32}{3}\zeta_2 + \frac{1024}{9}\zeta_3 - \frac{196}{3}\zeta_4 \right] + \frac{55}{243} - \frac{10045}{81}\zeta_3 - \frac{16}{9}\zeta_2\zeta_3 + \frac{2624}{27}\zeta_4 - \frac{176}{9}\zeta_5 \right. \\
& \left. - 8B_4 \right] + C_F^2 T_F \left[ L_Q^3 6 + L_M^2 L_Q 6 + L_M^2 \left[ -10 + \frac{32}{3}\zeta_3 \right] + L_Q^2 \left[ -48 + \frac{32}{3}\zeta_3 \right] \right. \\
& \left. + L_M L_Q 2 + L_M \left[ -5 - \frac{112}{9}\zeta_3 - \frac{136}{3}\zeta_4 \right] + L_Q \left[ \frac{368}{3} + \frac{64}{3}\zeta_2 - \frac{1616}{9}\zeta_3 \right. \right. \\
& \left. \left. + \frac{392}{3}\zeta_4 \right] - \frac{2039}{18} + \frac{13682}{81}\zeta_3 + \frac{32}{9}\zeta_2\zeta_3 - \frac{3304}{27}\zeta_4 + 16B_4 + \frac{352}{9}\zeta_5 \right] \\
& + C_F T_F^2 \left[ L_M^3 \frac{32}{9} + L_Q^3 \frac{16}{9} + L_M^2 \frac{8}{9} - L_Q^2 \frac{152}{9} + L_M \frac{496}{27} + L_Q \frac{1624}{27} - \frac{3658}{243} \right. \\
& \left. + \frac{224}{9}\zeta_3 \right] + C_F N_F T_F^2 \left[ L_M^3 \frac{16}{9} + L_Q^3 \frac{32}{9} - L_Q^2 \frac{304}{9} + L_M \frac{700}{27} + L_Q \frac{3248}{27} + \frac{4732}{243} \right. \\
& \left. - \frac{128}{9}\zeta_3 \right] \Big) + C_A C_F T_F \left[ L_M^3 \frac{88}{27}(x+1) + L_Q^3 \frac{176}{27}(x+1) + L_M^2 \left[ -\frac{4}{9}(83x-37) \right. \right. \\
& \left. \left. + \frac{32}{3}(x+1)H_0 + \frac{32}{3} \frac{x^2+1}{x+1} [H_{0,-1} - H_{-1}H_0] + \frac{16}{3} \frac{x}{x+1} [2\zeta_2 - H_0^2] \right] \right]
\end{aligned}$$

$$\begin{aligned}
& +L_Q^2 \left[ -\frac{4}{27}(865x+109) - \frac{256}{9}(x+1)H_0 + \frac{32}{3} \frac{x^2+1}{x+1} [H_{0,-1} - H_{-1}H_0] \right. \\
& - \frac{176}{9}(x+1)H_1 + \frac{16}{3} \frac{x}{x+1} [2\zeta_2 - H_0^2] \left. \right] + L_M \left[ -\frac{4}{81}(4577x-4267) \right. \\
& - \frac{16}{27}(29x-109)H_0 + \frac{4}{9} \frac{19x^2+4x+25}{x+1} H_0^2 - \frac{32}{9} \frac{x}{x+1} H_0^3 + \left( \frac{32}{3}(x-1) \right. \\
& + 8(x+1)H_0^2 \left. \right) H_1 + \frac{128}{9} \frac{4x^2+3x+4}{x+1} H_{0,-1} + \left( -\frac{16}{3} - 16H_0 \right) (x+1)H_{0,1} \\
& + \left( -\frac{128}{9} \frac{4x^2+3x+4}{x+1} H_0 + \frac{16}{3} \frac{x^2+1}{x+1} [4H_{0,1} - H_0^2] \right) H_{-1} + \frac{64}{3} \frac{x}{x+1} H_{0,0,1} \\
& + \frac{32}{3} \frac{x^2+1}{x+1} [H_{0,0,-1} - 2H_{0,-1,1} - 2H_{0,1,-1}] + \left( \frac{16}{9} \frac{3x^2+14x-9}{x+1} \right. \\
& - \frac{64}{3} \frac{x^2+1}{x+1} H_{-1} + \frac{16}{3} \frac{3x^2+4x+3}{x+1} H_0 \left. \right) \zeta_2 - 32 \frac{x^2+3x+1}{x+1} \zeta_3 \left. \right] \\
& + L_Q \left[ \frac{4}{81}(12329x-577) + \left( \frac{64}{27} \frac{181x^2+239x+49}{x+1} - \frac{32}{3} \frac{(x-1)^2}{x+1} H_{-1}^2 \right. \right. \\
& + \frac{32}{9} \frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x} H_{-1} \left. \right) H_0 + \left( -\frac{8}{9} \frac{12x^3-21x^2-77x-24}{x+1} \right. \\
& + \frac{16}{3} \frac{5x^2-2x+5}{x+1} H_{-1} \left. \right) H_0^2 + \frac{80}{9} \frac{x}{x+1} H_0^3 + \left( \frac{8}{27}(703x+253) - \frac{8}{3}(x-3)H_0^2 \right. \\
& + \frac{352}{9}(x+1)H_0 \left. \right) H_1 + \left( \frac{176}{9}(x+1) - \frac{16}{3}(x+1)H_0 \right) H_1^2 + \left( \frac{64}{3} \frac{3x+1}{x+1} H_0 \right. \\
& - \frac{32}{9} \frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x} + \frac{64}{3} \frac{(x-1)^2}{x+1} H_{-1} \left. \right) H_{0,-1} + \left( \frac{208}{9}(x+1) \right. \\
& + \frac{16}{3}(x-3)H_0 + \frac{64}{3}(x+1)H_1 \left. \right) H_{0,1} + \frac{32}{3}(x+3)H_{0,0,1} - 32(x+1)H_{0,1,1} \\
& - \frac{64}{3} \frac{(x-1)^2}{x+1} H_{0,-1,-1} - \frac{32}{3} \frac{5x^2+10x+9}{x+1} H_{0,0,-1} + \left( -\frac{64}{3}(x+2)H_1 \right. \\
& + \frac{16}{9} \frac{12x^3-23x^2-72x-17}{x+1} - \frac{32}{3} \frac{(x-1)^2}{x+1} H_{-1} - \frac{16}{3} \frac{3x^2+8x+3}{x+1} H_0 \left. \right) \zeta_2 \\
& + \frac{64}{3} \frac{3x^2+3x+2}{x+1} \zeta_3 \left. \right] - \frac{2}{729}(108295x-86681) + \left( -\frac{4}{81}(995x-2807) \right. \\
& - \frac{32}{81} \frac{199x^2+174x+199}{x+1} H_{-1} + \frac{32}{9}(x+1)H_{-1}^2 - \frac{64}{27} \frac{x^2+1}{x+1} H_{-1}^3 \left. \right) H_0 \\
& + \left( \frac{4}{81} \frac{253x^2+391x+586}{x+1} - \frac{16}{27} \frac{19x^2+18x+19}{x+1} H_{-1} + \frac{16}{9} \frac{x^2+1}{x+1} H_{-1}^2 \right) H_0^2 \\
& + \left( \frac{8}{81} \frac{22x^2+7x+25}{x+1} - \frac{32}{27} \frac{x^2+1}{x+1} H_{-1} \right) H_0^3 - \frac{16}{27} \frac{x}{x+1} H_0^4 + \left( \frac{8}{9}(9x+4)H_0 \right. \\
& - \frac{8}{27}(65x-29) + \frac{8}{9}(14x+3)H_0^2 + \frac{56}{27}(x+1)H_0^3 \left. \right) H_1 + \left( -\frac{4}{9}(43x-46) \right. \\
& - \frac{8}{9}(2x+5)H_0 - \frac{4}{9}(x+1)H_0^2 \left. \right) H_1^2 + \frac{32}{27}(x+1)H_0H_1^3 + \left( -\frac{64}{9}(x+1)H_{-1} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{32}{81} \frac{199x^2 + 174x + 199}{x+1} + \frac{64}{9} \frac{x^2 + 1}{x+1} H_{-1}^2 \Big) H_{0,-1} + \left( \frac{256}{27} \frac{4x^2 + 3x + 4}{x+1} H_{-1} \right. \\
& - \frac{8}{27} (143x + 2) - \frac{16}{9} (13x + 6) H_0 - \frac{8}{3} (x+1) H_0^2 + \left( \frac{8}{9} (11x + 20) \right. \\
& + \frac{64}{9} (x+1) H_0 \Big) H_1 - \frac{64}{9} (x+1) H_1^2 \Big) H_{0,1} + \frac{16}{9} (7x+1) H_{0,1}^2 + \left( \frac{64}{9} (x+1) \right. \\
& - \frac{128}{9} \frac{x^2 + 1}{x+1} H_{-1} \Big) H_{0,-1,-1} - \frac{256}{27} \frac{4x^2 + 3x + 4}{x+1} H_{0,-1,1} + \left( -\frac{64}{9} \frac{x^2 + 1}{x+1} H_{-1} \right. \\
& + \frac{32}{27} \frac{19x^2 + 18x + 19}{x+1} \Big) H_{0,0,-1} + \left( \frac{8}{27} \frac{9x^2 + 101x + 12}{x+1} + \frac{64}{9} \frac{x^2 + 1}{x+1} H_{-1} \right. \\
& - \frac{16}{9} (7x+1) H_0 - \frac{160}{9} (x+1) H_1 \Big) H_{0,0,1} - \frac{256}{27} \frac{4x^2 + 3x + 4}{x+1} H_{0,1,-1} \\
& + \left( -\frac{16}{9} (x+7) - \frac{128}{9} \frac{x^2 + 1}{x+1} H_{-1} - \frac{16}{9} (11x+5) H_0 + 16(x+1) H_1 \right) H_{0,1,1} \\
& + \frac{64}{9} \frac{x^2 + 1}{x+1} \left[ 2H_{0,-1,-1,-1} + 2H_{0,-1,1,1} + H_{0,0,-1,-1} - H_{0,0,-1,1} + H_{0,0,0,-1} \right. \\
& - H_{0,0,1,-1} + 2H_{0,1,-1,1} + 2H_{0,1,1,-1} \Big] + \frac{64}{9} \frac{5x^2 + 6x - 1}{x+1} H_{0,0,0,1} \\
& - \frac{16}{9} \frac{x^2 - 2x - 11}{x+1} H_{0,0,1,1} - \frac{112}{9} (x+1) H_{0,1,1,1} + \left( \frac{16}{81} \frac{174x^2 + 209x - 189}{x+1} \right. \\
& - \frac{32}{27} \frac{29x^2 + 18x + 29}{x+1} H_{-1} + \left( -\frac{8}{9} (3x+14) + \frac{80}{9} (x+1) H_0 \right) H_1 \\
& + \left( \frac{8}{27} \frac{63x^2 + 29x + 6}{x+1} - \frac{32}{9} \frac{x^2 + 1}{x+1} H_{-1} \right) H_0 - \frac{32}{9} \frac{x^2 + 1}{x+1} H_{-1}^2 + \frac{64}{9} (x+1) H_1^2 \\
& + \frac{8}{9} \frac{3x^2 + 8x + 9}{x+1} H_0^2 - \frac{16}{9} (7x+1) H_{0,1} \Big) \zeta_2 + \left( \frac{2}{27} \frac{497x^2 + 1102x + 1085}{x+1} \right. \\
& + \frac{128}{9} \frac{x^2 + 1}{x+1} H_{-1} + \frac{32}{9} \frac{6x^2 + 4x - 3}{x+1} H_0 + \frac{16}{9} (x+1) H_1 \Big) \zeta_3 + \frac{16}{3} (x+1) B_4 \\
& - \frac{8}{3} \frac{36x^2 + 51x + 22}{x+1} \zeta_4 \Big] + C_F^2 T_F \left[ [L_Q^3 + L_M^2 L_Q] \left[ -\frac{8}{3} (x+5) + \frac{32}{3} (x+1) H_1 \right. \right. \\
& + 8(x+1) H_0 \Big] + L_M^2 \left[ 28(2x-1) + \left( -\frac{8}{3} (11x+5) + \frac{64}{3} \frac{x^2 + 1}{x+1} H_{-1} \right) H_0 \right. \\
& - \frac{4}{3} \frac{9x^2 + 10x + 9}{x+1} H_0^2 + \left( -\frac{16}{3} (2x+1) - \frac{64}{3} (x+1) H_0 \right) H_1 - \frac{8}{3} (x+1) H_{0,1} \\
& - 8(x+1) H_1^2 - \frac{64}{3} \frac{x^2 + 1}{x+1} H_{0,-1} + \frac{8}{3} \frac{9x^2 + 10x + 9}{x+1} \zeta_2 \Big] + L_Q^2 \left[ \frac{4}{9} (161x + 130) \right. \\
& + \left( -\frac{16}{3} (15x+4) + \frac{64}{3} \frac{x^2 + 1}{x+1} H_{-1} \right) H_0 - 24(x+1) H_1^2 + \left( -\frac{16}{9} (50x+17) \right. \\
& - \frac{160}{3} (x+1) H_0 \Big) H_1 - \frac{4}{3} \frac{21x^2 + 34x + 21}{x+1} H_0^2 + \frac{8}{3} \frac{23x^2 + 38x + 23}{x+1} \zeta_2 \\
& - \frac{64}{3} \frac{x^2 + 1}{x+1} H_{0,-1} - 8(x+1) H_{0,1} \Big] + L_M L_Q \left[ \frac{4}{9} (19x - 85) + \frac{8}{3} (13x+1) H_0 \right.
\end{aligned}$$

$$\begin{aligned}
& +8(x+1)H_0^2 + \left( \frac{128}{9}(4x+1) + \frac{32}{3}(x+1)H_0 \right) H_1 - \frac{32}{3}(x+1)\zeta_2 \Big] \\
& + L_M \left[ -\frac{4}{9}(337x+235)H_0 - \frac{4}{9} \frac{195x^2+238x+123}{x+1} H_0^2 - \frac{32}{9} \frac{3x^2+4x+3}{x+1} H_0^3 \right. \\
& + \left( -\frac{4}{9}(287x-113) - \frac{224}{9}(5x+2)H_0 - \frac{80}{3}(x+1)H_0^2 \right) H_1 + \frac{4}{3}(178x-125) \\
& + \left( -\frac{16}{3}(7x+3) - \frac{16}{3}(x+1)H_0 \right) H_1^2 + \left( \frac{184}{9}(x+1) + \frac{32}{3}(x+1)H_0 \right) H_{0,1} \\
& - \frac{256}{9} \frac{4x^2+3x+4}{x+1} H_{0,-1} + \frac{16}{3} \frac{3x^2-2x+3}{x+1} H_{0,0,1} + \left( \frac{256}{9} \frac{4x^2+3x+4}{x+1} H_0 \right. \\
& + \left. \frac{32}{3} \frac{x^2+1}{x+1} [H_0^2 - 4H_{0,1}] \right) H_{-1} + \frac{64}{3} \frac{x^2+1}{x+1} [2H_{0,-1,1} - H_{0,0,-1} + 2H_{0,1,-1}] \\
& + \left( \frac{8}{9} \frac{117x^2+118x+81}{x+1} + \frac{128}{3} \frac{x^2+1}{x+1} H_{-1} + \frac{16}{3} \frac{(x+3)(3x+1)}{x+1} H_0 \right. \\
& + \left. \frac{32}{3}(x+1)H_1 \right) \zeta_2 + \frac{16}{3} \frac{x^2+14x+1}{x+1} \zeta_3 \Big] + L_Q \left[ -\frac{8}{27}(557x+652) \right. \\
& + \left( \frac{8}{9} \frac{115x^2+99x+32}{x+1} - \frac{64}{9} \frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x} H_{-1} \right. \\
& + \left. \frac{64}{3} \frac{(x-1)^2}{x+1} H_{-1}^2 \right) H_0 + \frac{32}{9} \frac{9x^2+13x+9}{x+1} H_0^3 + \left( -\frac{32}{3} \frac{5x^2-2x+5}{x+1} H_{-1} \right. \\
& + \left. \frac{4}{9} \frac{48x^3+519x^2+706x+315}{x+1} \right) H_0^2 + \left( \frac{8}{27}(908x-19) + \frac{16}{9}(169x+97)H_0 \right. \\
& + \left. \frac{32}{3}(7x+5)H_0^2 \right) H_1 + \left( \frac{32}{3}(13x+6) + 64(x+1)H_0 \right) H_1^2 + \frac{160}{9}(x+1)H_1^3 \\
& + \left( \frac{64}{9}(13x+1) + \frac{16}{3}(x+9)H_0 - \frac{32}{3}(x+1)H_1 \right) H_{0,1} + \left( -\frac{128}{3} \frac{3x+1}{x+1} H_0 \right. \\
& + \left. \frac{64}{9} \frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x} - \frac{128}{3} \frac{(x-1)^2}{x+1} H_{-1} \right) H_{0,-1} \\
& + \frac{128}{3} \frac{(x-1)^2}{x+1} H_{0,-1,-1} + \frac{64}{3} \frac{5x^2+10x+9}{x+1} H_{0,0,-1} + \frac{16}{3}(5x-3)H_{0,0,1} \\
& + 48(x+1)H_{0,1,1} + \left( -\frac{16}{9} \frac{24x^3+245x^2+318x+137}{x+1} + \frac{64}{3} \frac{(x-1)^2}{x+1} H_{-1} \right. \\
& - \left. \frac{16}{3} \frac{35x^2+66x+35}{x+1} H_0 - \frac{32}{3}(9x+5)H_1 \right) \zeta_2 - \frac{32}{3} \frac{21x^2+30x+17}{x+1} \zeta_3 \Big] \\
& + \frac{1}{27}(12332x-4905) + \left( -\frac{64}{9}(x+1)H_{-1}^2 + \frac{64}{81} \frac{199x^2+174x+199}{x+1} H_{-1} \right. \\
& + \frac{1}{81}(-10999x-8399) + \frac{128}{27} \frac{x^2+1}{x+1} H_{-1}^3 \Big) H_0 + \left( \frac{32}{27} \frac{19x^2+18x+19}{x+1} H_{-1} \right. \\
& - \left. \frac{32}{9} \frac{x^2+1}{x+1} H_{-1}^2 - \frac{2}{81} \frac{4179x^2+5255x+2868}{x+1} \right) H_0^2 + \left( \frac{64}{27} \frac{x^2+1}{x+1} H_{-1} \right. \\
& - \left. \frac{10}{81} \frac{177x^2+218x+105}{x+1} \right) H_0^3 - \frac{1}{27} \frac{51x^2+70x+51}{x+1} H_0^4 + \left( -\frac{152}{27}(x+1)H_0^3 \right.
\end{aligned}$$



$$\begin{aligned}
& + \frac{1}{27}(-3457x + 1951) - \frac{16}{81}(593x + 335)H_0 - \frac{8}{27}(146x + 71)H_0^2 \Big) H_1 \\
& + \left( -\frac{8}{9}(3x + 55) - \frac{8}{9}(9x + 1)H_0 + \frac{4}{9}(x + 1)H_0^2 \right) H_1^2 - \frac{64}{27}(x + 1)H_0H_1^3 \\
& + \left( -\frac{64}{81}\frac{199x^2 + 174x + 199}{x + 1} + \frac{128}{9}(x + 1)H_{-1} - \frac{128}{9}\frac{x^2 + 1}{x + 1}H_{-1}^2 \right) H_{0,-1} \\
& + \left( \frac{4}{27}(251x + 407) + \frac{16}{27}(10x + 43)H_0 + \frac{16}{3}(x + 1)H_0^2 + \left( \frac{64}{9}(x - 1) \right. \right. \\
& \left. \left. - \frac{80}{9}(x + 1)H_0 \right) H_1 - \frac{512}{27}\frac{4x^2 + 3x + 4}{x + 1}H_{-1} + \frac{64}{9}(x + 1)H_1^2 \right) H_{0,1} \\
& + \frac{8}{9}(x + 1)H_{0,1}^2 + \frac{512}{27}\frac{4x^2 + 3x + 4}{x + 1}H_{0,-1,1} + \left( \frac{256}{9}\frac{x^2 + 1}{x + 1}H_{-1} \right. \\
& \left. - \frac{128}{9}(x + 1) \right) H_{0,-1,-1} + \left( -\frac{64}{27}\frac{19x^2 + 18x + 19}{x + 1} + \frac{128}{9}\frac{x^2 + 1}{x + 1}H_{-1} \right) H_{0,0,-1} \\
& + \left( \frac{4}{27}\frac{357x^2 + 130x + 93}{x + 1} - \frac{128}{9}\frac{x^2 + 1}{x + 1}H_{-1} + \frac{64}{9}(x + 1)[2H_1 - H_0] \right) H_{0,0,1} \\
& + \frac{512}{27}\frac{4x^2 + 3x + 4}{x + 1}H_{0,1,-1} + \left( -\frac{32}{9}(13x + 1) + \frac{16}{3}(x + 1)[H_0 - 2H_1] \right. \\
& \left. + \frac{256}{9}\frac{x^2 + 1}{x + 1}H_{-1} \right) H_{0,1,1} + \frac{128}{9}\frac{x^2 + 1}{x + 1}[H_{0,0,-1,1} - 2H_{0,-1,-1,-1} - 2H_{0,-1,1,1} \\
& - H_{0,0,-1,-1} - H_{0,0,0,-1} + H_{0,0,1,-1} - 2H_{0,1,-1,1} - 2H_{0,1,1,-1}] \\
& + \frac{8}{9}\frac{21x^2 + 10x + 21}{x + 1}H_{0,0,0,1} - \frac{32}{9}\frac{7x^2 + 6x + 7}{x + 1}H_{0,0,1,1} + \left( \frac{64}{9}\frac{x^2 + 1}{x + 1}H_{-1}^2 \right. \\
& \left. + \frac{4}{81}\frac{1619x^2 + 1338x + 1511}{x + 1} + \left( \frac{4}{27}\frac{147x^2 + 298x - 9}{x + 1} + \frac{64}{9}\frac{x^2 + 1}{x + 1}H_{-1} \right) H_0 \right. \\
& \left. + \frac{64}{27}\frac{29x^2 + 18x + 29}{x + 1}H_{-1} + \frac{4}{9}\frac{(x + 5)(5x + 1)}{x + 1}H_0^2 - \frac{64}{9}(x + 1)H_1^2 \right. \\
& \left. + \left( \frac{16}{9}(x + 9) - \frac{16}{3}(x + 1)H_0 \right) H_1 + \frac{16}{9}(x + 1)H_{0,1} \right) \zeta_2 + \left( -\frac{80}{9}(x + 1)H_1 \right. \\
& \left. - \frac{8}{27}\frac{235x^2 + 404x + 409}{x + 1} - \frac{256}{9}\frac{x^2 + 1}{x + 1}H_{-1} + \frac{8}{9}\frac{15x^2 + 22x + 15}{x + 1}H_0 \right) \zeta_3 \\
& + \frac{4}{9}\frac{131x^2 + 178x + 131}{x + 1}\zeta_4 - \frac{32}{3}(x + 1)B_4 \Big] + C_F T_F^2 \left[ -L_M^3 \frac{64}{27}(x + 1) \right. \\
& \left. - L_Q^3 \frac{32}{27}(x + 1) + L_M^2 \left[ -\frac{32}{27}(11x - 1) - \frac{32}{9}(x + 1)H_0 \right] + L_Q^2 \left[ \frac{32}{27}(14x + 5) \right. \right. \\
& \left. \left. + \frac{64}{9}(x + 1)H_0 + \frac{32}{9}(x + 1)H_1 \right] - L_M \frac{992}{81}(x + 1) + L_Q \left[ -\frac{32}{81}(187x + 16) \right. \right. \\
& \left. \left. - \frac{64}{27}(28x + 13)H_0 - \frac{32}{3}(x + 1)H_0^2 + \left( -\frac{64}{27}(14x + 5) - \frac{64}{9}(x + 1)H_0 \right) H_1 \right. \right. \\
& \left. \left. - \frac{32}{9}(x + 1)H_1^2 - \frac{64}{9}(x + 1)H_{0,1} + \frac{128}{9}(x + 1)\zeta_2 \right] + \frac{16}{729}(431x + 323) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{81}(6x-7)H_0 + \frac{16}{81}(11x-1)H_0^2 + \frac{16}{81}(x+1)H_0^3 - \frac{448}{27}(x+1)\zeta_3 \Big] \\
& + C_F N_F T_F^2 \left[ -[L_M^3 + 2L_Q^3] \frac{32}{27}(x+1) + L_Q^2 \left[ \frac{64}{27}(14x+5) + \frac{128}{9}(x+1)H_0 \right. \right. \\
& + \frac{64}{9}(x+1)H_1 \Big] + L_M \left[ \frac{32}{81}(5x-73) + \frac{32}{27}(11x-1)H_0 + \frac{16}{9}(x+1)H_0^2 \right] \\
& + L_Q \left[ -\frac{64}{81}(187x+16) - \frac{128}{27}(28x+13)H_0 + \frac{64}{9}(x+1)[4\zeta_2 - 2H_{0,1} - H_1^2] \right. \\
& - \frac{64}{3}(x+1)H_0^2 + \left( -\frac{128}{9}(x+1)H_0 - \frac{128}{27}(14x+5) \right) H_1 \Big] + \frac{32}{81}(x+1)H_0^3 \\
& - \frac{64}{729}(161x+215) + \frac{128}{81}(6x-7)H_0 + \frac{32}{81}(11x-1)H_0^2 + \frac{256}{27}(x+1)\zeta_3 \Big] \\
& \left. + \hat{C}_{q,g1}^{\text{NS},(3)}(N_F) \right\}. \tag{2.61}
\end{aligned}$$

Again, we used the short hand notation  $H_{\bar{a}}(x) \equiv H_{\bar{a}}$  also here. The transformation of the Wilson coefficient to the  $\overline{\text{MS}}$  scheme for the heavy quark mass affects the massive OME at 3-loops and was given in Ref. [10]; the terms are the same in the unpolarized and polarized case.

The non-singlet contributions to the structure function  $g_2(x, Q^2)$  can be obtained via the Wandzura-Wilczek relation [15]

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2), \tag{2.62}$$

where both structure functions refer to the twist-2 contributions. This relation is implied by a relation of the OMEs in the light-cone expansion, cf. [39]. The relation has also been proven in the covariant parton model in Refs. [40–42]. For gluonic initial states, it was derived in [43]. Eq. (2.62) also holds including target mass corrections [44, 45] and finite light quark contributions [45]. Furthermore, it holds in non-forward [46] and diffractive scattering, including target mass corrections [47, 48].

### 3 Numerical Results

In what follows, we will choose the factorization and renormalization scale  $\mu^2 = Q^2$ . We first study the behaviour of the massive and massless Wilson coefficients in the small and large  $x$  region and then give numerical illustrations in the whole  $x$ -region.

At small  $x$ , the pure massive Wilson coefficient behaves like

$$L_{q,g1}^{\text{h,NS}}(N_F+1) - \hat{C}_{q,g1}^{\text{NS},(3)}(N_F) \propto a_s^2 4C_F T_F \ln^2(x) + a_s^3 \left[ \frac{16}{27} C_A C_F T_F - \frac{5}{9} C_F^2 T_F \right] \ln^4(x), \tag{3.1}$$

while in the region  $x \rightarrow 1$  one obtains

$$L_{q,g1}^{\text{h,NS}}(N_F+1) - \hat{C}_{q,g1}^{\text{NS},(3)}(N_F) \propto a_s^2 C_F T_F \frac{8}{3} \left( \frac{\ln^2(1-x)}{1-x} \right)_+ + a_s^3 C_F^2 T_F \left[ 16 \ln^2 \left( \frac{Q^2}{m^2} \right) \right]$$

$$+\frac{160}{3}\ln^2\left(\frac{Q^2}{m^2}\right)+\frac{448}{9}\left]\left(\frac{\ln^2(1-x)}{1-x}\right)\right)_+.$$
(3.2)

There is a term  $\propto \ln^3(1-x)/(1-x)$  at  $O(\ln(Q^2/\mu^2))$ , being of relevance for different choices of the factorization scale.

The above results can be compared with the case of the massless Wilson coefficient

$$\hat{C}_{q,g_1}^{\text{NS},(2)}(N_F) \propto a_s^2 \frac{10}{3} C_F T_F \ln^2(x) \quad (3.3)$$

$$\hat{C}_{q,g_1}^{\text{NS},(3)}(N_F) \propto a_s^3 \left[ \frac{92}{27} C_F C_A T_F - \frac{31}{9} C_F^2 T_F \right] \ln^4(x) \quad (3.4)$$

$$\hat{C}_{q,g_1}^{\text{NS},(2)}(N_F) \propto a_s^2 \frac{8}{3} C_F T_F \left( \frac{\ln^2(1-x)}{1-x} \right)_+ \quad (3.5)$$

$$\hat{C}_{q,g_1}^{\text{NS},(3)}(N_F) \propto a_s^3 \frac{80}{9} C_F^2 T_F \left( \frac{\ln^4(1-x)}{1-x} \right)_+. \quad (3.6)$$

The small  $x$  behaviour can be compared with leading order predictions for the non-singlet evolution kernel in Refs. [49, 50]. Indeed both the massive and massless contributions follow the principle pattern  $\sim c_k a_s^{k+1} \ln^{2k}(x)$ . However, as is well known [49], less singular terms widely cancel the numerical effect of these leading terms. For the large  $x$  terms the massless terms exhibit a stronger soft singularity than the massive ones.

In the following numerical illustrations we use the polarized parton distributions of Ref. [8], which are of next-to-leading order (NLO), since no next-to-next-to-leading order (NNLO) data analysis based on the anomalous dimensions calculated in Ref. [51] has been performed yet. The values of  $\alpha_s$  correspond to those of the unpolarized NNLO analysis [52]. The heavy and light flavor Wilson coefficients being discussed in the following are given in Eqs. (2.2) and (2.3).

In Figure 1, the 2- and 3-loop heavy flavor corrections to the non-singlet term of the structure function  $xg_1(x, Q^2)$  are calculated in the case of charm, assuming  $m_c = 1.59$  GeV [53], using the formula for the Wilson coefficient Eq. (2.61), and setting  $\mu^2 = Q^2$ . With growing  $Q^2$ , the distribution diminishes at larger values of  $x$  and grows towards medium values. The  $O(\alpha_s^3)$  corrections lead to stronger effects if compared to those at  $O(\alpha_s^2)$ . We have applied the asymptotic Wilson coefficients for all the  $Q^2$  values given here, which only holds for values  $Q^2/m^2 \gtrsim 10$ . For the heavy quark distributions we formally show also the result at  $Q^2 = 4$  GeV<sup>2</sup>, outside this region, indicated by dotted ( $O(a_s^2)$ ) and dash-dotted lines ( $O(a_s^3)$ ).

Figure 2 shows the effect of the Wilson coefficients comparing the contributions from  $O(\alpha_s^0)$  to  $O(\alpha_s^3)$  at  $Q^2 = 4$  GeV<sup>2</sup> as an example, where a depletion is obtained with growing order. The 3-loop light flavor contributions to  $xg_1(x, Q^2)$  ( $N_F = 3$ ) are illustrated in Figure 3. Here the evolution is strengthened by growing  $Q^2$  in the large  $x$  region and depleted for lower values of  $Q^2$ , considering only the effects due to the Wilson coefficient.

In Figures 4 and 5 we illustrate the ratio of the flavor non-singlet charm corrections to those by the light quarks given in Eq. (2.3) up to  $O(\alpha_s^2)$  and  $O(\alpha_s^3)$ , respectively. At  $O(\alpha_s^2)$  the effect is of  $O(1\%)$  and below, for the lower scales  $Q^2$ , but higher values are obtained for very large scales as  $Q^2 \simeq 1000$  GeV<sup>2</sup> in the region  $x \sim 0.003$ . A qualitatively similar picture is obtained including the  $O(\alpha_s^3)$  corrections. The effect on the ratio  $g_1^{\text{heavy}}/g_1^{\text{light}}|_{\text{NS}}$  is about doubled. To resolve relative effects of  $O(2\%)$  requires higher luminosities than available in present day experiments. They may become available in the planned experiments at a future EIC [54].

Figure 6 shows the 2- and 3-loop charm flavor non-singlet contributions to the structure function  $xg_2(x, Q^2)$  according to the Wandzura-Wilczek relation (2.62) implying the oscillatory

behaviour. In size these effects are comparable to those of the structure function  $xg_1(x, Q^2)$  shown in Figure 1. With growing  $Q^2$  the effects become somewhat smaller. In Figure 7 we show the corresponding massless contributions to the structure function  $g_2(x, Q^2)$  at  $Q^2 = 4 \text{ GeV}^2$  for the different orders in  $a_s$ , which slightly diminish adding higher order contributions. Taking into account the  $O(a_s^3)$  corrections, the light flavor corrections to  $g_2(x, Q^2)$  (1.5, 2.62) grow somewhat in size with larger values of  $Q^2$ , see Figure 8. Similar to the case of the structure function  $xg_1$  the  $O(a_s^3)$  charm flavor non-singlet corrections to the structure function  $xg_2(x, Q^2)$  amount to  $O(1\%)$ .

## 4 The Bjorken Sum Rule

The polarized Bjorken sum rule [55] refers to the first moment of the flavor non-singlet combination

$$\int_0^1 dx [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)] = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{\text{BJ}}(\hat{a}_s), \quad (4.1)$$

with  $g_{A,V}$  the neutron decay constants,  $g_A/g_V \approx -1.2767 \pm 0.0016$  [56] and  $\hat{a}_s = \alpha_s/\pi$ . The 1- [57], 2- [58], 3- [30] and 4-loop QCD corrections [31] in the massless case are given by

$$\begin{aligned} C_{\text{BJ}}(\hat{a}_s), &= 1 - \hat{a}_s + \hat{a}_s^2(-4.58333 + 0.33333N_F) + \hat{a}_s^3(-41.4399 + 7.60729N_F - 0.17747N_F^2) \\ &+ \hat{a}_s^4(-479.448 + 123.472N_F - 7.69747N_F^2 + 0.10374N_F^3), \end{aligned} \quad (4.2)$$

choosing the renormalization scale  $\mu^2 = Q^2$ , cf. [28] for  $SU(3)_c$ . Here  $N_F$  denotes the number of active light flavors. The expression for general color factors was given in Ref. [31].

For the asymptotic massive corrections (2.2) only the first moments of the massless Wilson coefficients  $\hat{C}_{q,g_1}^{(2,3),\text{NS}}(N_F)$  contribute, since the first moments of the massive non-singlet OMEs vanish due to fermion number conservation, a property holding even at higher order. Therefore, any new heavy quark changes Eq. (4.2) by a shift in  $N_F \rightarrow N_F + 1$  only, for the asymptotic corrections. Different results are obtained in the tagged flavor case [5, 7] at  $O(\alpha_s^2)$ , where no inclusive structure functions are considered. Corresponding power corrections were derived in [59, 60].

## 5 Conclusions

We calculated the heavy flavor non-singlet Wilson coefficients of the polarized inclusive structure function  $g_1(x, Q^2)$  to  $O(\alpha_s^3)$  in the asymptotic region  $Q^2 \gg m^2$ . The first contributions of this kind are of  $O(\alpha_s^2)$ . In the case of twist-2 operators the corresponding contributions to the structure function  $g_2(x, Q^2)$  can be obtained using the Wandzura-Wilczek relation (2.62) [15], cf. [39–42, 45]. The asymptotic Wilson coefficient is obtained by using the factorization formula [5], Eq. (2.2), based on the massive OME [10] and the massless Wilson coefficient [29] to 3-loop order. The heavy flavor Wilson coefficient can be thoroughly represented by nested harmonic sums in Mellin- $N$  space and by harmonic polylogarithms in  $x$ -space. We presented numerical results corresponding to the charge weighted polarized parton contributions  $\propto \Delta f(x, Q^2) + \Delta \bar{f}(x, Q^2)$ , cf. (1.5), referring to the polarized parton distribution functions at NLO [8] for an illustration. Comparing with the corresponding massless cases the heavy flavor corrections in case of charm are of  $O(1 - 2\%)$ , requiring high luminosity experiments to be resolved, which are planned for the future electron-ion collider EIC [54]. We also considered the contribution

of the asymptotic Wilson coefficient to the polarized Bjorken sum-rule. Due to fermion number conservation for the massive flavor non-singlet OME in all orders in  $\alpha_s$ , only the first moment of the massless Wilson coefficient contributes and the effect of each heavy flavor results in a shift of  $N_F$  by one unit in the expression for the massless polarized Bjorken sum-rule. The results of the present calculation could be easily applied to derive the asymptotic heavy flavor corrections to the neutral current structure function  $xG_3$ , [61]. However, the corresponding massless Wilson coefficient to 3-loop order has not been calculated yet.

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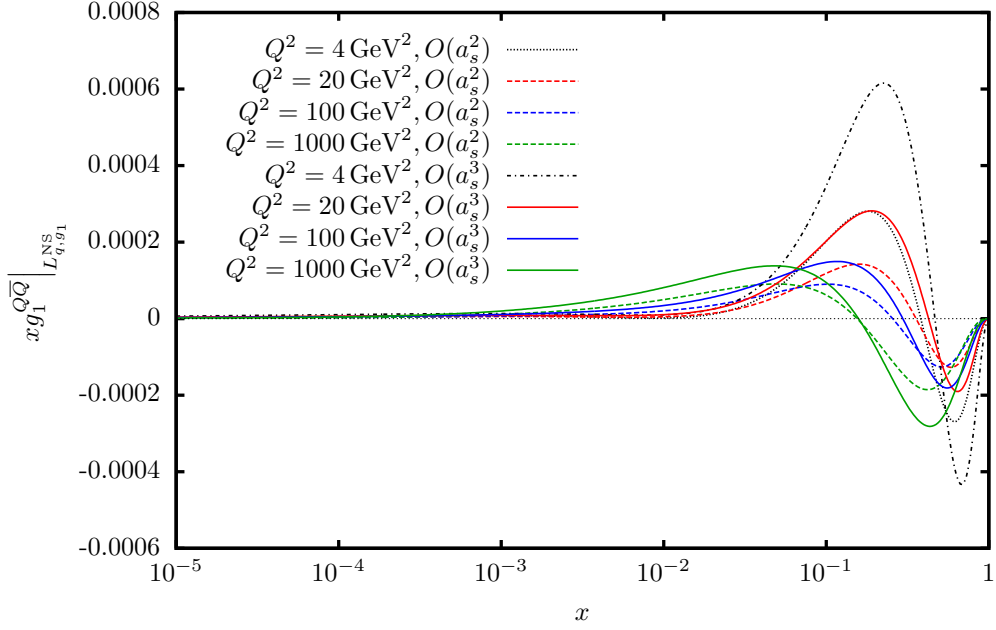


Figure 1: The 2- and 3-loop non-singlet charm contributions to the structure function  $xg_1(x, Q^2)$  by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme for  $m_c = 1.59$  GeV. Here we used the value of  $\alpha_s(M_Z^2) = 0.1132$  and the NLO parton distribution [8] as reference. Figures 2–8 below are calculated using the same setting.

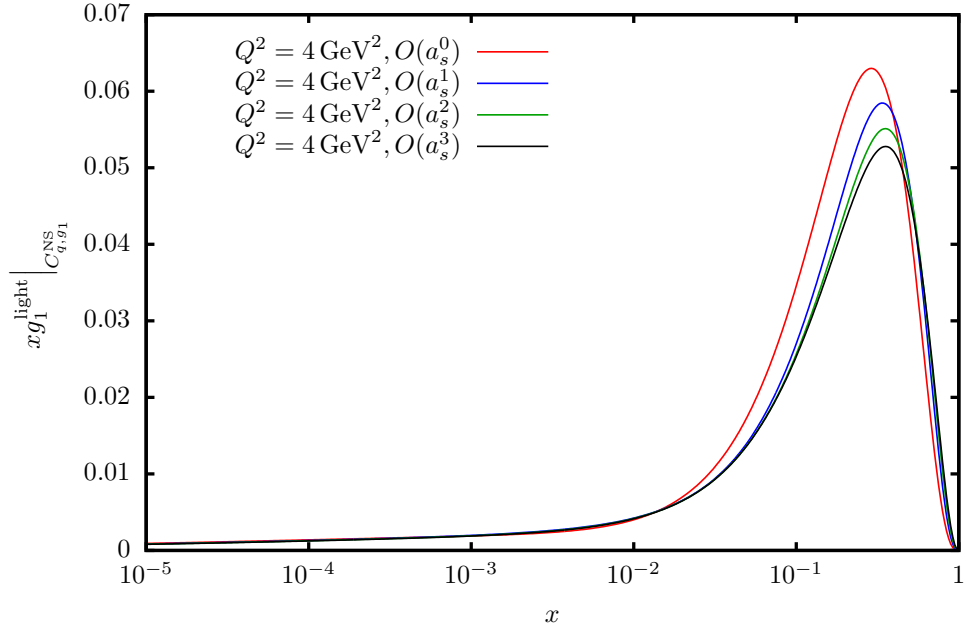


Figure 2: The light flavor contributions ( $N_F = 3$ ) to the non-singlet charm contributions to the structure function  $xg_1(x, Q^2)$  at  $Q^2 = 4$  GeV $^2$  illustrating the contributions for the different orders in  $a_s$ .

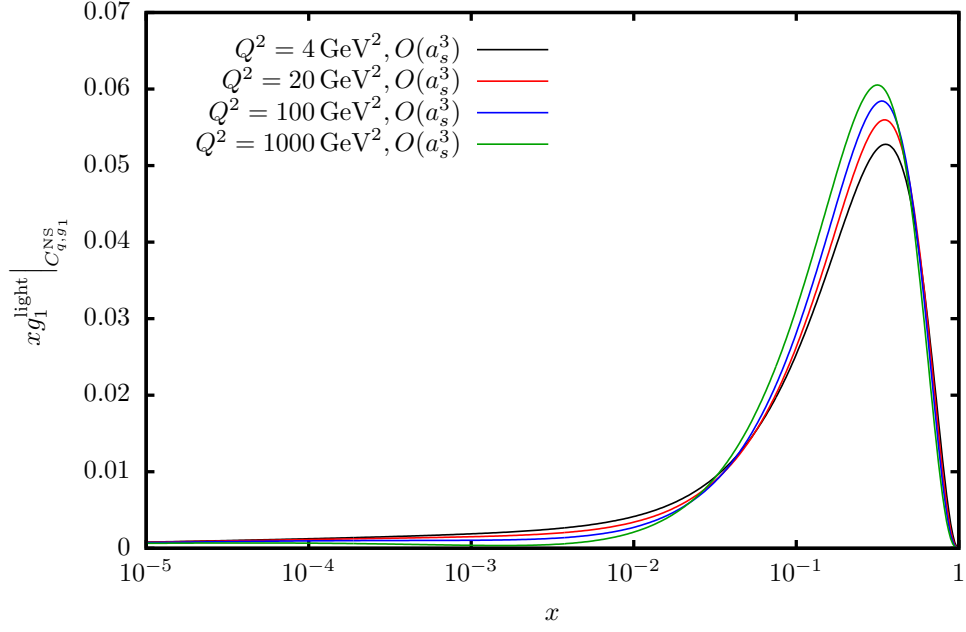


Figure 3: The light flavor contributions ( $N_F = 3$ ) to the non-singlet charm contributions to the structure function  $xg_1(x, Q^2)$  at  $O(a_s^3)$  for different values of  $Q^2$ .

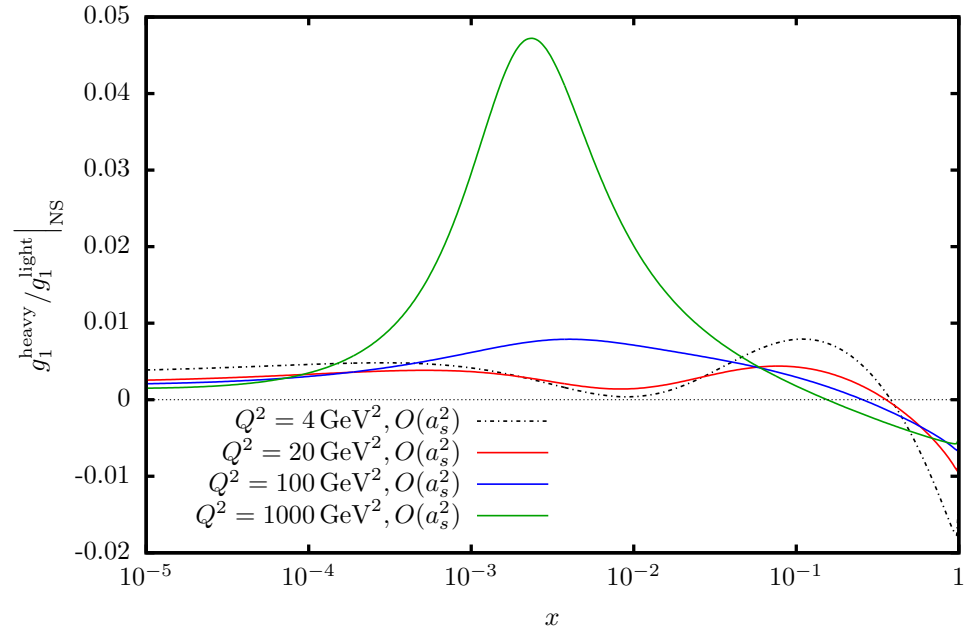


Figure 4: The ratio  $g_1^{\text{charm}}/g_1^{\text{light}}|_{\text{NS}}$  in the non-singlet case at  $O(a_s^2)$  for different values of  $Q^2$ .

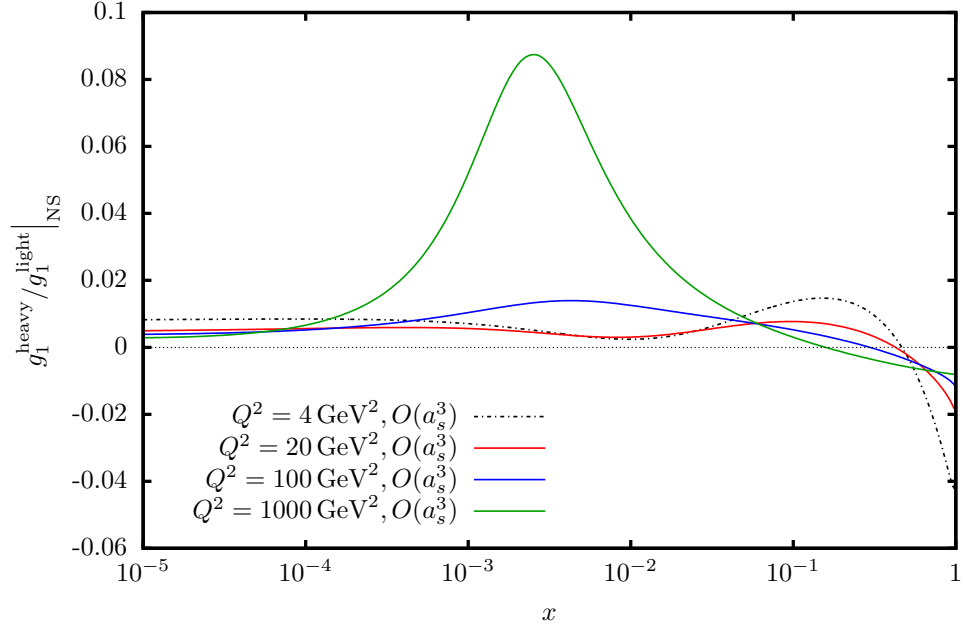


Figure 5: The ratio  $g_1^{\text{charm}}/g_1^{\text{light}}|_{\text{NS}}$  in the non-singlet case at  $O(a_s^3)$  for different values of  $Q^2$ .

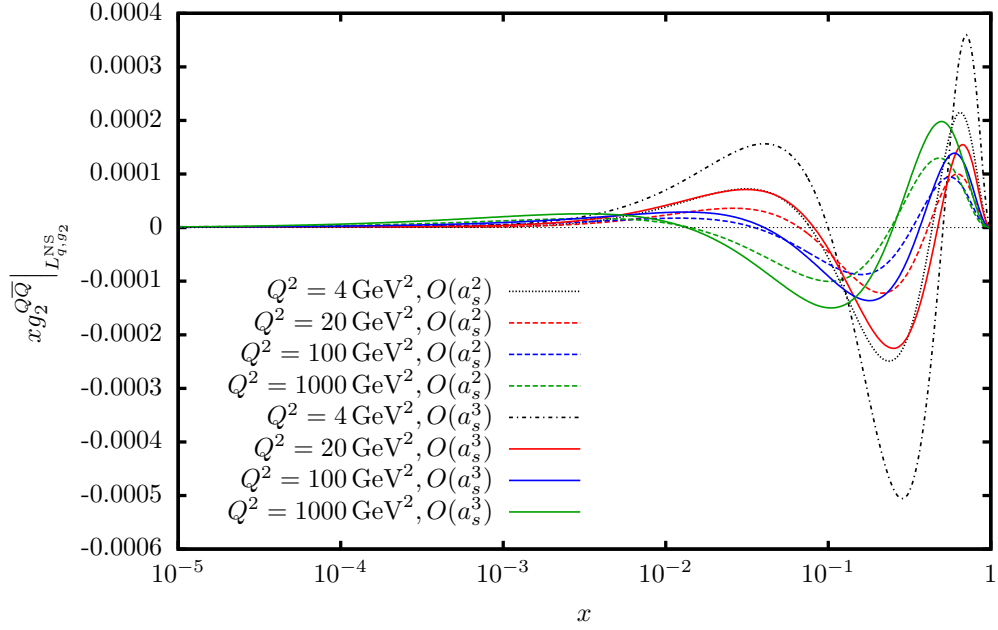


Figure 6: The 2- and 3-loop non-singlet charm contributions to the twist 2 contributions of the structure function  $xg_2(x, Q^2)$  by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme.



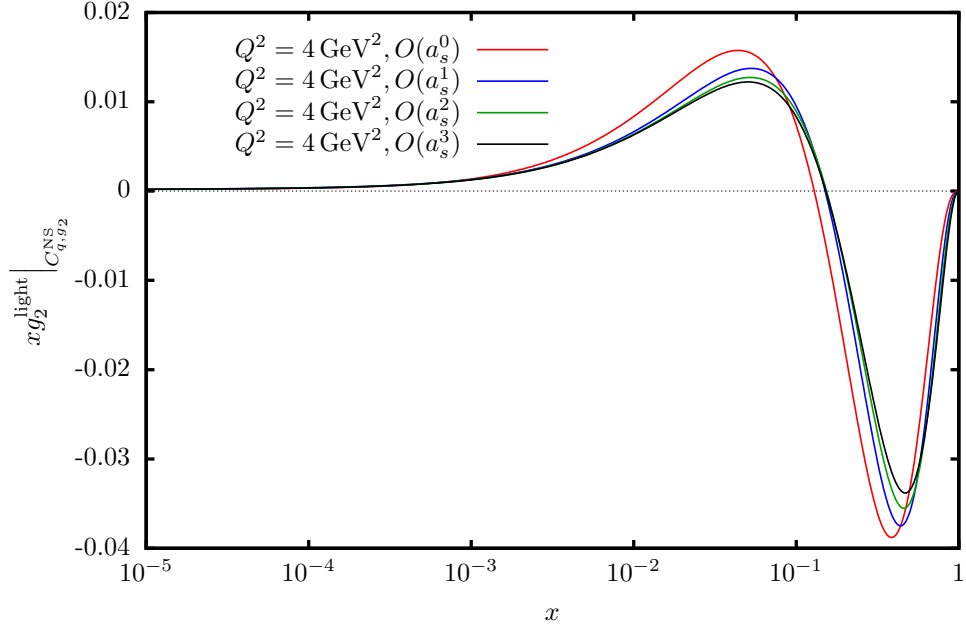


Figure 7: The light flavor contributions ( $N_F = 3$ ) to the non-singlet charm contributions to the structure function  $xg_2(x, Q^2)$  at  $Q^2 = 4 \text{ GeV}^2$  illustrating the contributions for the different orders in  $a_s$ .

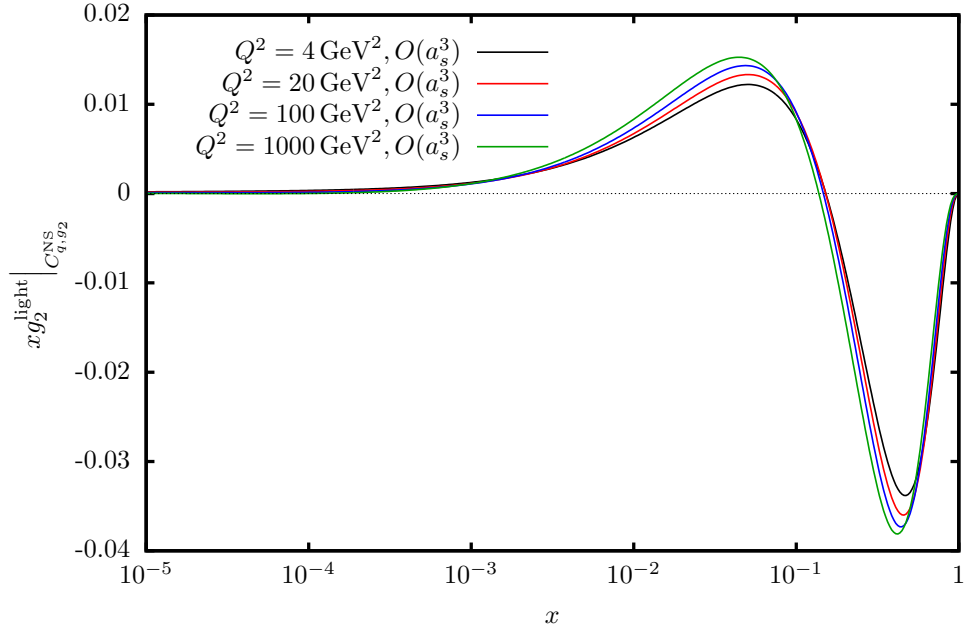


Figure 8: The light flavor contributions ( $N_F = 3$ ) to the non-singlet charm contributions to the structure function  $xg_2(x, Q^2)$  at  $O(a_s^3)$  for different values of  $Q^2$ .

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